F-RD

An Embedded Triply Periodic Minimal Surface (TPMS)

Space group symbol: $F\frac{4}{m}\bar{3}\frac{2}{m}$

Genus of the associated Riemann surface: 6

This infinite surface, which was discovered in 1969 [Schoen 1970], has the same symmetries as a cubic close-packed assembly of congruent spheres. F-RD is an *embedded* surface, *i.e.,* it has no self-intersections.

Every straight line lying in a minimal surface is an axis of 2-fold rotational symmetry (Schwarz's reflection principle [Schwarz 1890]). All of the examples of embedded TPMS known before 1970 contain straight lines, which are called *linear asymptotics*. Hence for every such surface, the two disjoint interpenetrating *labyrinths* into which R^3 is partitioned are congruent. Such surfaces are conventionally called *balanced* [Fischer and Koch 1986]. F-RD contains no in-surface straight lines. Its two labyrinths are not congruent, and it is therefore called *unbalanced.*

The relation between F-RD and a face-centered cubic (f.c.c.) lattice packing of spheres can be described picturesquely in terms of the following 'ball-and-spoke' model of the f.c.c. crystal structure:

> Transform each ball in the structure into a hollow sphere, in which twelve small circular holes are cut out. Each hole in a given sphere is centered on a line from the sphere center to the center of one of the twelve nearest neighbor spheres. Join each pair of nearest neighbor spheres by a hollow cylindrical tube ('handle'), smoothly attached at each of its ends to the rim of a sphere hole. Finally, let this infinite periodic structure be transformed into a soap film surface that is allowed to relax into equilibrium, with the same air pressure on both sides of the surface, without changing either its topology or symmetry. The mean curvature of this final soap film surface, which is F-RD, is zero everywhere.

The labyrinth obtained from the f.c.c. network in the procedure described above is conveniently represented by its *skeletal graph* [Schoen 1970], an infinite symmetric graph whose nodes are f.c.c. lattice points, each joined by an edge to its twelve nearest neighbor lattice points. The 'F' in F-RD is the name of this skeletal graph. Its dual, the skeletal graph of the labyrinth on the opposite side of the surface, is not a symmetric graph. Its nodes and edges are the vertices and edges, respectively, of an infinite packing of *rhombic dodecahedra*—whence its name: 'RD'. Two-thirds of the nodes of the RD graph are of degree four, and one-third are of degree eight.

Let us call a surface of non-zero constant mean curvature an H-surface, and let H^* denote the mean curvature of the surface. Numerical calculations [Anderson *et al* 1990] provide empirical evidence that the area of F-RD is a maximum with respect to all H-surfaces that belong to the same family as F-RD. Since all of these surfaces have the same symmetry and the same topology as F-RD, they are described by the same pair of skeletal graphs as F-RD. According to Anderson *et al*:

> 'The fact that a local maximum in area occurs at $H^* = 0$ [both for the F-RD family and also for each of the other families of H-surfaces studied by Anderson *et al*] is not predicted by any known theorem. Schwarz showed that under the orthogonality boundary conditions, the second variation of the area is negative for a minimal surface bounded by the planes of a tetrahedron, but this only means that some normal perturbation which preserves the orthogonality boundary conditions decreases the area, and does not say anything specifically about those perturbations with constant mean curvature.'

Hildebrandt, Grüter, and Nitsche [Hildebrandt 1985] [Grüter *et al* 1986] proved that only in the case of periodic surfaces of constant mean curvature is R^3 partitioned into fixed volume fractions such that the surface area is stationary with respect to all area perturbations.

The topological complexity of a periodic minimal surface is defined by the genus of one lattice fundamental region, regarded as embedded in the 3-torus T^3. The genus is equal to that of the Riemann surface defined by mapping the surface normal onto the unit sphere (Gauss map). A lattice fundamental region of F-RD is defined by an assembly of 48 *Flächenstücke* (smallest repeating units), each related to four others by reflection in the planes of its four curved boundary edges.

It was shown by Weierstrass [Weierstrass 1866] that each rectangular coordinate x, y, and z of a point on a minimal surface is a harmonic function of the complex variable ω:

$$(x, y, z) = Re\ e^{i\theta} \int^{\omega} \left(1 - \omega'^2,\ i\left(1 + \omega'^2\right),\ 2\omega'\right) R(\omega')\, d\omega',$$

where R is an analytic function.

In 1934, Stessmann [Stessmann 1934] carried out an incomplete study of the Weierstrass function R for the self-intersecting surface A(F-RD), which is the *adjoint* of F-RD. This Weierstrass function also provides the solution for F-RD itself (solutions for any two adjoint surfaces differ only in the value of $\theta : \theta = 0$ for one and $\pi/2$ for the other). Stessmann treated only A(F-RD), not considering either the local or global properties of F-RD. He apparently did not realize that F-RD is an embedded TPMS, which makes it considerably more interesting than A(F-RD). That F-RD is embedded was first recognized by the author in 1969 [Schoen 1970].

The *Flächenstuck* of A(F-RD) is a minimal surface bounded by a skew quadrilateral with *straight* edges. This polygon is one of six proved by Schoenfliess [Schoenfliess 1890] to be the only skew quadrilaterals spanned by minimal surfaces that generate TPMS by half-turn rotations about their edges. A fundamental relation between adjoint minimal surfaces implies that the *Flächenstuck* of F-RD is bounded by four *curved* edges, each of which is a segment of a plane line of curvature.

Two of the other five Schoenfliess quadrilaterals define Schwarz's D surface and its adjoint, Schwarz's P surface, respectively. Each of these surfaces is an embedded balanced TPMS of genus 3.

Of the three remaining Schoenfliess quadrilaterals, two define self-intersecting periodic surfaces whose respective adjoints are embedded TPMS. One of these is Neovius's surface C(P) (genus 9). The other, which was first identified by the author in 1969 [Schoen 1970] [Cvijovic *et al*], is called I-WP (genus 4). C(P) has the same in-surface straight lines as Schwarz's P surface and is therefore a balanced surface, but I-WP—like F-RD—contains no straight lines and is unbalanced. The sixth Schoenfliess quadrilateral has one 120° corner, which accounts for a branch point (self-intersection).

Fogden and Hyde [Fogden *et al* 1992] proved that for every TPMS of the 'regular' class, which is defined by the property that only locally equivalent flat points are superposed in the Gauss map, R can be expressed in the form of the following simple product:

$$R(\omega) = \exp(i\theta) \prod_{i=1}^{n} (\omega - \omega_i)^{-b_i/(b_i+1)};$$

n is the number of distinct stereographic images ω_i, in the complex plane, of flat-point normal-vectors in the Gauss map, b_i is the order of the flat point with normal vector image ω_i, and θ is the Bonnet angle of associativity [Bonnet 1853].

It is easily verified that F-RD is not a TPMS of the regular class. The Weierstrass function $R(\omega)$ for F-RD was recently derived by Fogden [Fogden 1992], who proved that $R(\omega)$ is defined by the equation

$$p_1^2 p_2 R^5 - 15\, p_1^2 R^3 + 5\, p_2 R^2 - 48 = 0;$$

p_1 is the Weierstrass polynomial $\omega\,(\omega^4 + 1)$ for I-WP and p_2 is the Weierstrass polynomial $\omega^8 - 14\omega^4 + 1$ for both D and P. The Weierstrass functions $R(\omega)$ for I-WP and for D and P are $[\omega(\omega+1)^{-2/3}]$ and $[\omega^8 - 14\omega^4 + 1]^{-1/2}$, respectively. It is not possible to solve Fogden's fifth-degree polynomial equation analytically for R; it must be solved numerically.

The occurrence of p_1 and p_2 in the coefficients of the equation for R for F-RD is a consequence of the following fact:

> in the neighborhood of each of its six flat points of degree 3, F-RD is asymptotically congruent to I-WP near any of I-WP's six flat points of degree 3, and
> in the neighborhood of each of its eight flat points of degree 2, F-RD is asymptotically congruent to both P and D near any of their eight flat points of degree 2.

Like F-RD, Neovius's surface C(P) (which is also a member of the irregular class), has six flat points of degree 3 and eight flat points of degree 2. Fogden [Fogden 1992] derived the following equation for its Weierstrass function $R(\omega)$:

$$\frac{1}{4}\, p_D p_I^4 R^8 - p_I^4 R^6 + 2\, p_D R^2 + 1 = 0\,;$$

p_D is the Weierstrass polynomial $3\omega^8 + 28\omega^6 - 14\omega^4 + 28\omega^2 + 3$ for P and D, and p_I is the Weierstrass polynomial $\omega^6 - 5\omega^4 - 5\omega^2 + 1$ for I-WP. This solution, as pointed out by Fogden, is equivalent to that derived by Neovius [Neovius 1883].

Smyth [Smyth 1984] provided the first complete proof that F-RD is embedded. He also proved the following remarkable result:

> **Theorem:** *Let M be the boundary of a tetrahedron in R^3. There exist exactly 3 stationary minimal surfaces of disk type with boundary on M having connected intersection with each of the faces of M; these are all imbedded, non-planar and free from interior branch points; in fact each is a graph. For all stationary minimal surfaces with boundary on M—whether of disk type or not—the ratio of perimeter to area is the same, namely, 2/r, where r is the inradius of M.*

The table below lists the value of dimensionless area $A/V^{2/3}$ per lattice fundamental region for all TPMS with a cubic space lattice for which data are available. The values of $A/V^{2/3}$ increase with genus, as expected. Except in the case of F-RD, the value of $A/V^{2/3}$ is known exactly. (For F-RD, it is only *conjectured* [Anderson *et al* 1990] that $A/V^{2/3} = 3K(k)/K(k')$, where $k^2 = 8(3)^{1/2}/[21 + 4(3)^{1/2}]$.)

TPMS	genus	$A/V^{2/3}$
P	3	~ 2.3451 (= $3K(1/2)K'(1/2)$ [a]
D	3	~ 2.4177 (= $(3/2^{2/3})\,K'(1/2)/K(1/2)$) [a]
G	3	~ 2.4533 (= $(3/4^{2/3})\,(K'(1/2)/K(1/2))\,[1+(K(1/2)K'(1/2))^2]$ [a]
I-WP	4	~ 2.7495 (= $2^{2/3}3^{1/2}$) [b]
F-RD	6	~ 3.0054 (= $3K/K'$) ($k^2 = 8(3)^{1/2}/[21 + 4(3)^{1/2}]$) [c]
C(P)	9	~ 3.5105 (= $3\,K'/K$) [a]

[a] [Schoen 1970]
[b] [Cvijovic *et al* 1994]
[c] [Anderson *et al* 1990]

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