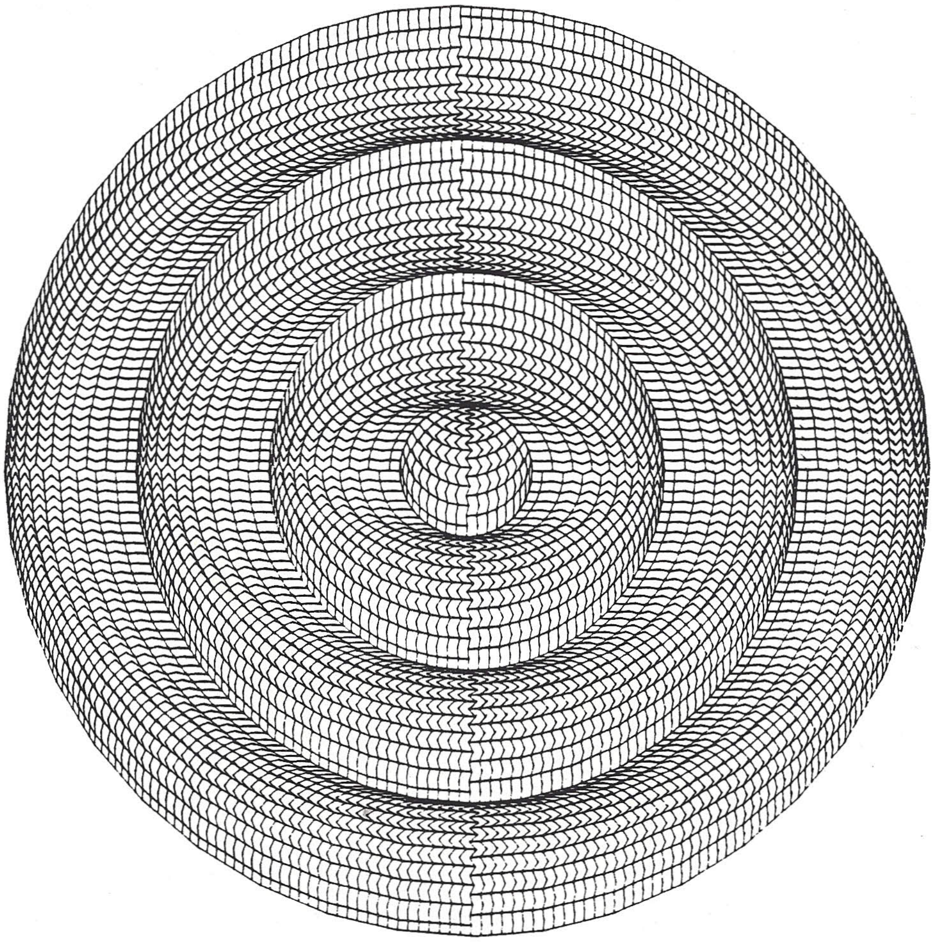
ROMBIX

Supplementary manual



The illustration on the cover shows 49 sets of ROMBIX-40 arranged in orderly concentric rings

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This four-color puzzle/game is made of injection-molded high-impact plastic.
It includes a tray and cover, plus an illustrated instruction sheet.

U. S. Patents Nos. 4,223,890 and 5,314,183
Japanese Patent No. 1659198

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FOREWORD

This potpourri is a provisional supplement to the instructions that accompany the commercial version of the puzzle/game called ROMBIX. A number of topics that are not included here, as well as expanded treatments of some of the included topics, are covered in a book about ROMBIX that I plan to publish in 1995.

The most widely known two-dimensional combinatorial dissection puzzles are polyominoes and their various close relatives. The variety in the shapes of the pieces in a set of n-ominoes is the result of combining n replicas of a single square module, edge-to-edge, in every possible way [Golomb 1965] [Golomb 1994] [Gardner 1966] [Klarner 1981] [Grünbaum and Shephard 1987] [Martin 1991]. ROMBIX is also a combinatorial dissection puzzle, but it differs from polyominoes in several fundamental ways. Its pieces are derived not from a single module but from n differently shaped modules, and these modules—which are rhombs—are combined only in pairs. This property, as well as the fact that some single rhombs are also included in each set, makes the behavior of ROMBIX—both as a puzzle and as a game—rather different from that of polyominoes.

The commercial version of ROMBIX is denoted here as ROMBIX-16, to distinguish it from ROMBIX-$2n$, the 'generic' set. The sixteen rombiks of ROMBIX-16 include four single rhombs (*keystones*) and twelve connected pairs of rhombs (*twins*). These sixteen pieces can be arranged in a variety of ways to tile the regular 16-gon, which is the only *convex* arena in which they can be arranged.

Some of the topics treated in this manual concern properties that are unique to ROMBIX-16, but I have searched also for properties of ROMBIX that depend on n in a systematic way. I have found, for example, that for certain values of n, the underlying set of rhombs ('SRI_{2n}') from which the ROMBIX-$2n$ set is constructed has special equi-partitioning and rearrangement properties that can be explained by the application of elementary number theory. Alan Shorb collaborated with me on these two problems [Schoen and Shorb 1994]. Equi-partioning is concerned with distributing the rhombs of SRI_{2n} among congruent convex shapes. Rearrangement means arranging the rhombs of SRI_{2n} to tile a convex region that is not congruent to the regular $2n$-gon. These topics are the subject of an article, written jointly with Alan Shorb, that will be published shortly. They are also treated in the forthcoming book about ROMBIX.

Another property of ROMBIX sets that I recently discovered—and proved by number-theoretic arguments—is described in §2.5:

Only if n-1 is prime can every pair of monochrome subsets form matched ladders.

I hope that readers will write to me about their own discoveries concerning ROMBIX. I learned recently that in 1993, Michael Reid found an elegant solution to the Segregated Color Circle Tiling problem for ROMBIX-16. This topic is described in the instructions in the ROMBIX-16 package. In Reid's tiling, no rombik of the awkward subset is incident on the boundary of the tiling arena. Please let me know if you find a comparable solution for any of the other three subsets.

I hope that readers do not object to my casual use of terms like *tile*, *tiling*, and *Circle Tiling*. I use the word *tiling* for finite regions—whether simply-connected or not—as well as for infinite regions, even though I recognize that it is customary to reserve this word for infinite regions only. I use the term *Circle Tiling* to denote an edge-to-edge arrangement of the set of rombiks inside the regular $2n$-gon arena.

I wish to express my sincere thanks to Donald Coxeter, Martin Gardner, and Solomon Golomb, who first kindled my interest in the geometry of the plane. I have probably learned as much mathematics from reading Coxeter's books and articles as I have from any other single source. As almost everyone knows, reading Gardner's columns and books is the most enjoyable of all known ways to learn mathematics. Many puzzle and game problems for ROMBIX were inspired by concepts originally developed by Golomb for polyominoes. To cite just one example, consider the *superposition* problems [Golomb 1994 p. 12], which provided the model for the topics I have treated in §4.16-4.18 of this manual. The first edition of Golomb's *Polyominoes* [Golomb 1965] was a joy to read. The second edition [Golomb 1994] is even more satisfying!

It was Roger Penrose's discoveries of the aperiodic tilings that bear his name [Penrose 1974] [Penrose 1978] that first led me to become interested in plane tilings by rhombs. I first learned about Penrose tilings by reading Martin Gardner's account in his *Scientific American* column [Gardner 1977].

I thank Richard Guy for his description of ROMBIX (then called 'Cyclotome') in Volume 2 of *Winning Ways for your mathematical plays* [Berlekamp, Conway, and Guy 1982].

I regularly consult *Tilings and Patterns*, by Branko Grünbaum and Geoffrey Shephard [Grünbaum and Shephard 1987], for both inspiration and information.

Ira Gessel and Don Redmond have both proved theorems related to combinatorial properties of sets of rhombs. These theorems are not included in this manual, but they will be described in the forthcoming book about ROMBIX.

Bill and Ruth Perk discovered that four identical ROMBIX sets can be arranged in a large Circle Tiling that has c4 symmetry (*cf.* §4.4). This pattern was found by the Perks when they modified the transformation of the CRACKED EGG for n=8 into the Triangle Array (*cf.* §4.3).

Finally, I extend my warm thanks to Alan Shorb, whose mathematical insights and proofs have enriched my understanding of the subject of Ovals (*cf.* §3).

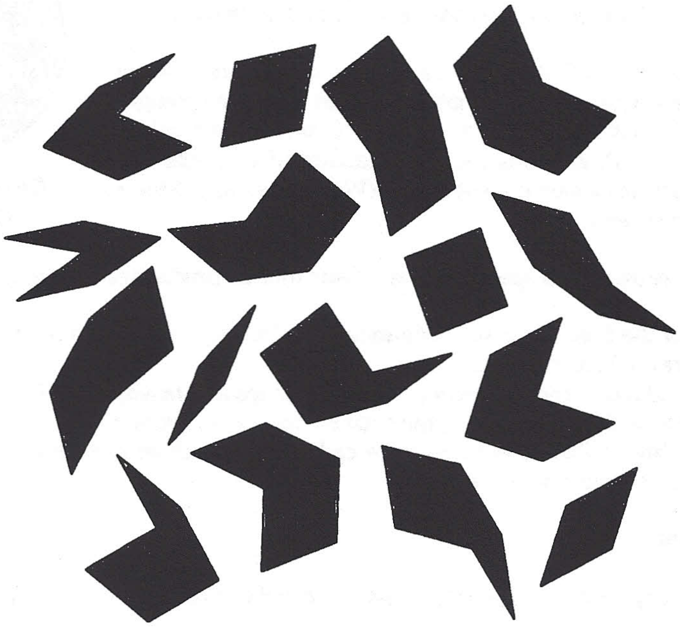
Alan Schoen

CONTENTS

			page
	FOREWORD		i
1	**INTRODUCTION**		1
	1.1	*Composition of the ROMBIX-16 set*	1
	1.2	*The CRACKED EGG and Monochrome Subsets*	2
	1.3	*Monochrome subsets all have the same area*	2
	1.4	*Composition of the ROMBIX-2n set*	2
2	**LADDERS**		3
	2.1	*Definition of ladder*	3
	2.2	*Symmetrical Ladders for ROMBIX-16*	5
	2.3	*The number of different shapes of monochrome ladders for ROMBIX-16*	5
	2.4	*The number of different shapes of monochrome ladders (even n)*	5
	2.5	*Matching Ladders (even n only)*	6
3	**OVALS**		9
	3.1	*Definition of Oval*	9
	3.2	*Monochrome 4-Ovals*	11
	3.3	*Hollowing Out the CRACKED EGG (tiling by rombiks)*	11
	3.4	*Shaving the Strawberry (tiling by rhombs)*	11
	3.5	*Ovals Embedded in Circle Tilings by rombiks*	12
	3.6	*Pied 4-Ovals*	12
	3.7	*The association of Ovals in conjugate pairs*	13
	3.8	*Strictly Convex Ovals and Stretched Ovals*	16

4	**MISCELLANY**		19
	4.1	*Chaotic Circle Tilings for ROMBIX-2n sets for n>>8*	19
	4.2	*A Macintosh program for drawing the CRACKED EGG*	19
	4.3	*The Triangle Array for ROMBIX-40*	20
	4.4	*Concentric Nested Ring tilings by ROMBIX-2n sets*	21
	4.5	*Chaotic Eccentric Nested Ring tilings by ROMBIX-2n sets*	22
	4.6	*Ten Tiling Tips*	26
	4.7	*Time required for a chaotic Circle Tiling*	28
	4.8	*Chipped Ovals and the tilings in which they are embedded*	30
	4.9	*Tiling the two-set RACETRACK with segregated colors*	30
	4.10	*Stars of order n tiled by two sets of ROMBIX-2n*	31
	4.11	*The tiling of Stars by the rhombs in two sets of* SRI_{2n}	32
	4.12	*The shape of the outline of a Star*	32
	4.13	*Embedded hexagons—an open problem*	33
	4.14	*French's Fences (or 'the Farm problem')*	33
	4.15	*The two families of quasi-uniform tilings*	34
	4.16	*Congruent Islands (ROMBIX-16)*	39
	4.17	*Symmetrical Islands (ROMBIX-16)*	40
	4.18	*Congruent Atolls (ROMBIX-16)*	40
	4.19	*LOOSE ENDS (ROMBIX-16)*	40
	BIBLIOGRAPHY		41
	NAME INDEX		42

1. INTRODUCTION



THE SIXTEEN ROMBIKS OF ROMBIX-16

1.1 Composition of the ROMBIX-16 set

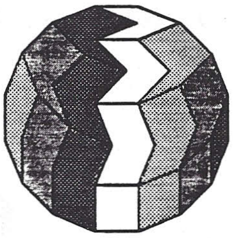
There are sixteen pieces in ROMBIX-16. Each is called a 'rombik'. Four of the rombiks are single rhombs and are called **keystones.** (A rhomb is a parallelogram whose sides are of equal length). The smaller face angles of the keystones are equal to 22.5° ($\pi/8$), 45° ($2\pi/8$), 67.5° ($3\pi/8$), and 90° ($4\pi/8$), respectively. The other twelve rombiks are called **twins**. Each twin is a *non-convex hexagon* formed by joining two single rhombs along a common edge. The edges of all rombiks are of the same length.

Verify that the twelve twin rombiks of ROMBIX-16 correspond to the twelve distinct ways in which rhombs congruent to the four keystone rhombs can be combined in pairs to form non-convex hexagons.

Nine of these twelve twin rombiks are *fraternal* twins: each is composed of two rhombs of different shape. The remaining three twins are *identical* twins: each is composed of two rhombs of the same shape, joined to make a chevron-shaped rombik.

The cavity in the ROMBIX-16 tray, which is in the shape of a regular sixteen-sided polygon, is called the *arena*. (The '16' in the name ROMBIX-16 refers to the number of sides of the arena. It's just a coincidence that the number of rombiks in this set is also sixteen.) The sixteen rombiks can be arranged inside the arena in a great variety of patterns called *Circle Tilings.* The exact number of Circle Tilings is unknown, but I conjecture that there are two thousand or more.

1.2 The CRACKED EGG and Monochrome Subsets



CRACKED EGG
ROMBIX-16)

The CRACKED EGG, shown at the right, is the only *orderly* Circle Tiling. All other Circle Tilings are defined as *chaotic*.

The rombiks in ROMBIX-16 are colored so that they define seven monochrome strips—or *ribbons*—when they are arranged to tile the CRACKED EGG. The ribbons come in four colors: green, yellow, red, and blue. Each collection of rombiks of a given color defines a *monochrome subset*. (We will usually refer to it simply as a *subset*.)

The blue subset has a special characteristic that distinguishes it from the other three:

> Each of the blue twins is an identical twin, while the twins in the other three subsets are all fraternal twins.
> Identical twins are *symmetrical*; fraternal twins are *unsymmetrical*.
> Because of their symmetry, the blue twins are less versatile than the twins in the other three subsets. We therefore call the blue subset *awkward* and each of the other three subsets *graceful*.

Verify that

(a) every twin rombik in the *awkward* subset is an *identical* twin;

(b) every twin rombik in each *graceful* subset is a *fraternal* twin.

1.3 Monochrome subsets all have the same area

Verify that the inventory of rhombs in each of the four subsets is the same, and that all four of the subsets therefore have the same area.

1.4 Composition of the set ROMBIX-2*n*

ROMBIX-16 is just one of a theoretically infinite number of ROMBIX sets. Each set is composed of one specimen of every concave twin that can be formed from a pair of the $\binom{n}{2}$ rhombs that tile the regular 2*n*-gon, plus one specimen of each of the different shapes of rhomb in this set.

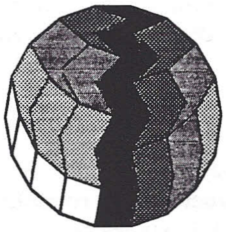
The *order* of a ROMBIX set is equal to *n*, which is equal to half the number of sides of the regular polygon that defines the arena for the set. There is a ROMBIX set of order *n* for every positive integer $n \geq 2$. For the trivial case $n=2$, ROMBIX-4 consists of only a single square. The interesting sets are those for which $n \geq 4$. ROMBIX-16 has been found to be the most versatile of all ROMBIX sets, but some sets of other orders have distinctive properties that are not discussed here.

The mathematical notation '⌊*number*⌋' means 'the integer part of *number*', *i.e.*, the integer that remains after the fractional part of *number* is thrown away. For example, $\lfloor n/2 \rfloor$ is equal to 4 for both n=8 and n=9. The smaller face angle of every rhomb in the rombiks of ROMBIX-2n is equal to $j\pi/n$ ($1 \le j \le \lfloor n/2 \rfloor$); j is called the **principal index** of the rhomb. Each of the $\lfloor (n+1)/2 \rfloor$ monochrome subsets is identified by its **subset number**. For even n, the subset number is equal to the principal index of the keystone in the subset. Subset numbering for odd n is defined below.

Verify that in ROMBIX-16, subset 1 is green, 2 is yellow, 3 is red, and 4 is blue.

The total number of rombiks in the set ROMBIX-2n is equal to $\lfloor n^2/4 \rfloor$. Of this number, $\lfloor n/2 \rfloor$ are keystones, and $\lfloor ((n-1)/2)^2 \rfloor$ are twins. Of the twins, $\lfloor (n-1)/2 \rfloor$ are identical, and $\lfloor ((n-2)/2)^2 \rfloor$ are fraternal.

For odd n, the keystones are sequestered into a single small subset—subset number $(n+1)/2$; the twin rombiks are distributed among the remaining $(n-1)/2$ subsets. This scheme is illustrated by the coloring of the CRACKED EGG for n=9, shown at the right. The area of the keystone subset is equal to exactly half the area of each of the twin subsets. Subset numbers increase from 1 to 4 with increasing shading. The ribbons on the right of the Crack are vertical, while those on the left are roughly horizontal.



CRACKED EGG
(ROMBIX-18)

The distribution of the colored ribbons in the CRACKED EGGS for n=8 and n=9 illustrates a general scheme for partitioning ROMBIX sets of any integer order n into monochrome subsets. This scheme has the following consequences:

For even n, there are $n/2$ subsets of equal area; each subset contains $n/2-1$ twins and one keystone.

For odd n, there are $(n-1)/2$ subsets of equal area, each of which contains $(n-1)/2$ twins. The $(n-1)/2$ keystones are sequestered into a single subset whose area is equal to half the area of each of the other subsets.

The distinction between graceful and awkward subsets holds for both odd and even n. For every order n, the subset whose twins are all identical twins is called awkward, while all other twin subsets are called graceful.

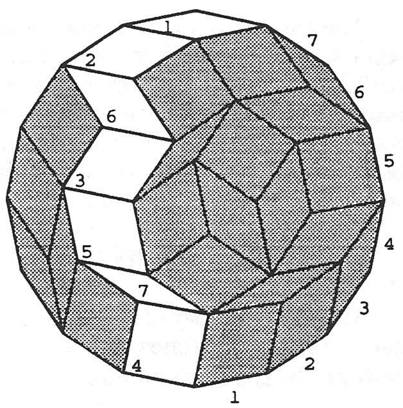
Why is it impossible to tile a regular polygon that has an *odd* number of sides with rhombs?

2. LADDERS

2.1 Definition of ladder

A **ladder** is a strip of rhombs or rombiks, with parallel *rungs*, that extends from one edge of a Circle Tiling of the regular 2n-gon arena to the opposite edge. It is comprised of n-1 rhombs. There are n ladders in every Circle Tiling of the arena.

In the chaotic Circle Tiling of ROMBIX-2n shown at the right, one ladder is shown in white. Observe that the indices of the slopes of the sides of the rhombs in this ladder assume each of n-1 possible integer values. Coxeter [Rouse Ball and Coxeter 1962] has proved that this property holds for every ladder, in Circle Tilings of every order n.

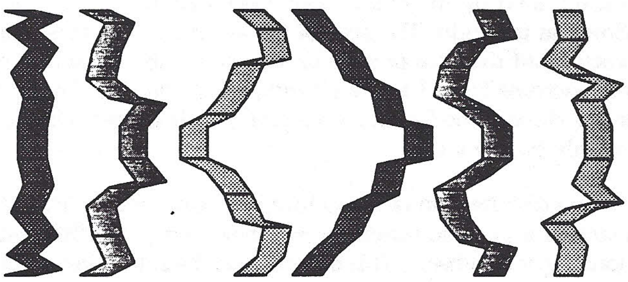


LADDER FOR n=8

Construct the CRACKED EGG Circle Tiling for ROMBIX-16. Examine the seven rhombs in each of the eight ladders, and verify that the slopes of the sides of these rhombs assume n-1 different values.

The six *symmetrical* ladders that can be constructed from rombiks of a single subset for n=14 are shown at the right.

(Why are there only six—not seven—symmetrical monochrome ladders for n=14?)



THE SIX SYMMETRICAL MONOCHROME LADDERS FOR n=14

Why is *reflection* symmetry the only allowed kind of symmetry for a ladder, whether monochrome or not?
For odd n, there are no symmetrical monochrome ladders. Why?

Mathematical aside:

For any even positive integer n, let s = the subset number for a graceful subset. Then

(a) if n-1 is an *odd prime* (*viz.*, n = 4, 6, 8, 12, 14, 18, 20, ...), the rombiks of every graceful subset s can be arranged to tile one *symmetrical* ladder;

(b) if n-1 is *composite* (*viz..*, n = 10, 16, 22, 26, 28, 29, ...), the rombiks of every graceful subset s can be arranged to tile one *symmetrical* ladder if and only if n-1 and $n/2 - s$ are relatively prime (*i.e.*, have no common divisor). Otherwise, no symmetrical ladder for subset s is possible.

These and other properties of ladders are proved in a technical note entitled Some combinatorial properties of heterosets of even order n [Schoen 1994]. This report is available, at no cost, on request from the author. Write to:

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Dept. of Electrical Engineering
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Carbondale, IL 62901.

In §2.5 (below), we make use of some of the properties of ladders that are proved in [Schoen 1994].)

2.2 Symmetrical ladders for ROMBIX-16

Construct a *symmetrical* ladder from the four rombiks of one of the three graceful subsets. Place this ladder inside the tray, and try to complete a Circle Tiling with the remaining twelve rombiks without modifying the symmetrical ladder. This problem has a solution for two of the three graceful subsets, but not for the third!

2.3 The number of different shapes of monochrome ladders for ROMBIX-16

Arrange the four rombiks of each graceful subset to form a ladder. There are 96 different shapes of ladders for each graceful subset.

Next, arrange the four rombiks of the awkward (blue) subset to form a ladder. Choose a shape that is different from that of the backbone ladder of the CRACKED EGG. There are altogether 48 different shapes of blue ladders.

2.4 The number of different shapes of monochrome ladders (even n)

Let $a(n)$ = the number of different shapes of monochrome ladders for the awkward subset, and

$g(n)$ = the number of different shapes of monochrome ladders for the graceful subset.

It is easily proved that $a(n)$ and $g(n)$ are given by the following expressions:

<u>n=4</u> $a(4) = 1;\ g(4) = 2.$

<u>n>4</u> $a(n) = 6 \cdot 8 \cdot 10 \cdots n\ ;$

$g(n) = 2\ a(n)\ .$

Table 2.4.1 shows the values of $g(n)$ and $a(n)$ for $4 \le n \le 30$.

Table 2.4.1
The number of distinct ladders which can be tiled by graceful subsets and by awkward subsets
$g(n)$ = the number of graceful subsets
$a(n)$ = the number of awkward subsets
$4 \le n \le 30$ (n even)

n	$g(n)$	$a(n)$
4	2	1
6	12	6
8	96	48
10	960	480
12	11,520	5,760
14	161,280	60,640
16	2,580,480	1,290,240
18	46,448,640	23,224,320
20	928,972,800	464,486,400
22	20,438,401,600	10,218,700,800
24	490,497,638,400	245,248,819,200
26	12,752,938,598,400	6,376,469,299,200
28	357,082,280,755,200	178,541,140,377,600
30	10,712,468,422,656,000	5,356,234,211,328,000

2.5 Matching ladders (even *n* only)

We now briefly summarize some features of the problem of *matching ladders*—constructing two different monochrome ladders which have the same overall shape. This problem is defined for *even n* only. **(Why is it not defined also for odd *n*?)**

We define the *orientation* of a rombik to be *face up* (U) if it is oriented in the same way as in the CRACKED EGG arranged with keystones on the right, as shown in the illustration for n=8 on p. 2, and *face down* (D) otherwise.

The necessary and sufficient conditions for a pair of subsets to form matched ladders are derived in [Schoen 1994]. It is proved there that iff n-1 is *prime,* then

(a) every pair of distinct subsets forms exactly one pair of matched ladders;

(b) the *orientation* of all the rombiks of a given subset in a matched ladder is the same—either face up (U) or face down (D). The orientations of the rombiks in two matched ladders are UU, UD, or DD, respectively.

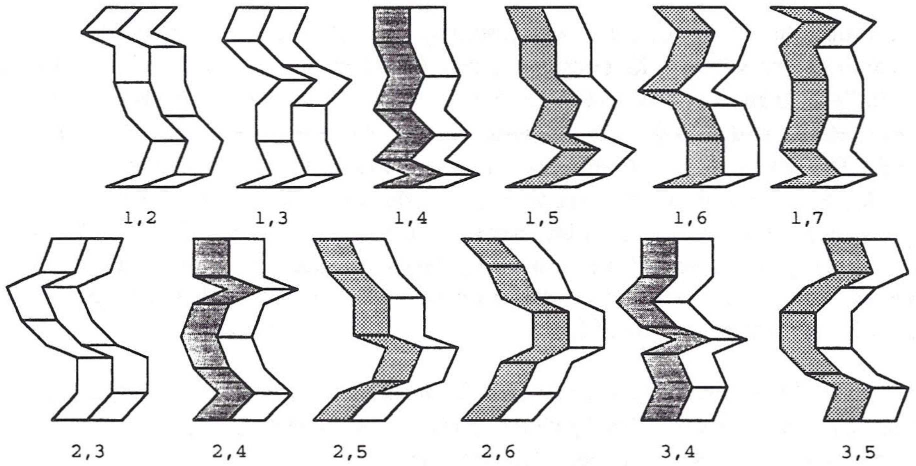
(c) two sets of the rombiks of each graceful subset can be arranged to form a pair of matched ladders, each of which is related to the other by reflection in the transverse midline of the ladder outline. The orientations of the rombiks in the two ladders are UD, respectively.

If n-1 is *composite,* then the ladder matching behavior of a pair of monochrome subsets depends on the *cycle distance* between the subset numbers for the pair of subsets [Schoen 1994]. The cycle distance D(j, k) between subsets j and k is defined as follows:

$$D(j,k) = |k-j| \quad \text{if } |k-j| < n/2, \text{ and}$$
$$= n-1-|k-j| \quad \text{if } |k-j| \geq n/2.$$

The cycle distance corresponds to the shorter of the two circular arcs that join points j and k in a *circle diagram.* Such a diagram consists of n-1 equally spaced points, on the boundary of a circle, that are labelled consecutively from 1 to n-1.

Ladder matching is illustrated below by the twelve possible pairs of matched ladders for n=8. The first of the two integers which label each pair is the index of the slope of the sides of the keystone in the *right*most subset of the pair. The second of the two integers is the index of the slope of the sides of the keystone in the *left*most subset of the pair. (The subset numbers for n=8 are 1, 2, 3, and 4. The labels 5, 6, and 7, which are the '4-complements' of subset numbers 3, 2, and 1, respectively, identify these same subsets. Subset 4 is its own 4-complement.)



THE MATCHED LADDER PAIRS FOR n=8
No shading: face up
Light shading: face down
Dark shading: awkward subset

(The descriptions 'face up' and 'face down' refer to the orientation of the rombiks in the Cracked Egg Circle Tiling.)

Finally, let us prove that for every pair of matched ladders, the keystones of the respective subsets lie at opposite ends of the ladder outline.

THEOREM 2.5.1
Let L_{AB} denote the outline of a ladder that can be tiled by either of two distinct monochrome subsets A and B. Let T_A be a tiling of L_{AB} by subset A and T_B a tiling of L_{AB} by subset B. Then

> **K_A, the keystone of A, occupies the rhombic site at one end of T_A, and**
> **K_B, the keystone of B, occupies the rhombic site at the opposite end of T_B.**

Proof

Suppose—contrary to the assertion of the theorem—that in the covering of L_{AB} by subset A in the tiling T_A, K_A occupies not a terminal rhombic site of L_{AB} but an interior rhombic site of L_{AB}. Because all of the other rombiks of subset A are twins, the two ladder segments of L_{AB} that lie on opposite sides of K_A, which we will call *end segments*, each contain an even number of rhombs. Let us call one of these end segments E_1 and the other E_2.

Consider how the rhombic site occupied by K_A in T_A is covered in the tiling T_B. This site cannot be covered by K_B, because K_A and K_B are not congruent. Therefore it is covered in T_B by a rhomb in one of the twins of subset B. That twin, which we will call τ, covers both K_A and an adjoining rhombic site that is in either E_1 or E_2. Let us say that it is in E_1. The covering in T_B of the other rhombic sites in E_1 by rombiks from B must include K_B, since the number of these sites is odd. But that implies that in T_B, E_2 is covered entirely by twins from B. That is impossible, since E_2 is covered in T_A entirely by twins from A, and there are no twins in B that are congruent to any of the twins in A. We conclude that K_A cannot cover an interior rhombic site of L_{AB}, and that it must therefore cover a terminal rhombic site of L_{AB}.

A similar argument applied to subset B implies that K_B covers a terminal rhombic site of L_{AB} in the tiling T_B. But K_B cannot cover the same end of L_{AB} as K_A, because it is not congruent to K_A. Hence it covers the opposite end.

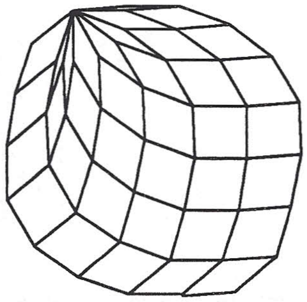
COROLLARY 2.5.1
No more than two monochrome subsets can tile the same ladder outline.

We leave the proof to the reader.

3. OVALS

3.1 Definition of Ovals

An Oval is a centro-symmetric convex polygon, with sides of unit length, each of whose turning angles is a positive integer multiple of π/n (integer $n \geq 2$). (The turning angle at a vertex of a polygon is the supplement of the interior face angle at that vertex.) Every Oval has an even number of sides, which are arranged in g parallel pairs. An Oval with $2g$ sides is called a 'g-Oval' and is determined by the values of n and g and by its Turning Angle Index Sequence ('TAIS'), which is a list of the turning angle indices for any consecutive set of g of its vertices.



n=45; g=9
TAIS = [9 8 2 3 7 6 4 5 1]

It is conjectured that for every positive integer $n \geq 2$, every g-Oval can be tiled by a subset of the rombiks in ROMBIX-$2n$. (Coxeter proved that every regular $2n$-gon can be tiled by $\binom{n}{2}$ rhombs whose sides are of the same length as those of the regular $2n$-gon [Ball and Coxeter 1962]. We call this set of rhombs the 'Standard Rhombic Inventory', or SRI_{2n}.) I have proved that every g-Oval can be tiled by a subset of SRI_{2n}. The smaller face angle of each rhomb in SRI_{2n} is equal to $k\pi/n$ ($k \in [1, \lfloor n/2 \rfloor]$); k is called the *principal index* of the rhomb. When n is odd, there are n specimens of each of the $(n\text{-}1)/2$ shapes of rhomb in SRI_{2n}. When n is even, there are n specimens of each of the $n/2-1$ non-square rhombs in SRI_{2n}, but there are only $n/2$ specimens of the square rhombs. The g-Oval illustrated above, for which n=45 and g=9, is shown tiled by 36 rhombs in the *STRAWBERRY* pattern. Note that for every integer n, $\binom{n}{2}$ is a triangular number. (For $n<2$, $\binom{n}{2}$ is defined to be equal to zero.)

The inventory of rhombs needed to tile an Oval can be derived from its TAIS by constructing the *Oval Index Triangle*. This construction is described below for the 9-Oval for n=45 that is shown above.

(a) Delete the first turning angle index from the TAIS, thereby obtaining the sequence of indices for the upper interior face angles of the rhombs in the *receptacle*—the cluster of rhombs that are incident on the *stem* vertex of the Oval. ('Receptacle' is the term used by botanists to denote the part of a plant that holds the fruit.) After deleting the first turning angle index from the TAIS of the Oval illustrated above, we obtain the sequence [8 2 3 7 6 4 5 1]. Let us call this sequence 'the truncated TAIS'.

(b) To form the first row of the Oval Index Triangle, enter the truncated TAIS.

8 2 3 7 6 4 5 1

(c) In the second row of the Oval Index Triangle, directly below each pair of consecutive indices in the first row, enter their sum.

```
8   2   3   7   6   4   5   1
 10   5  10  13  10   9   6
```

(d) In position k of the ith row ($i \geq 3$; $k = 1, 2, ..., g\text{-}i$), enter the sum of the indices in the kth sequence (counting from the left) of i consecutive indices in the first row. This kth entry in the ith row is also the sum of the two consecutive indices in the (i-1)th row that lie immediately above it, minus the index in the (i-2)th row that lies immediately above these two consecutive indices.

```
8   2   3   7   6   4   5   1
 10   5  10  13  10   9   6
   13  12  16  17  15  10
     20  18  20  22  16
       26  22  25  23
         30  27  26
           35  28
             36
```

(e) In each row, replace every index j that is greater than $\lfloor n/2 \rfloor$ by $n\text{-}j$. The Oval Index Triangle is now complete.

```
8   2   3   7   6   4   5   1
 10   5  10  13  10   9   6
   13  12  16  17  15  10
     20  18  20  22  16
       19  22  20  22
         15  18  19
           10  17
              9
```

(f) Count the number of occurrences v_q of the index q ($q = 1, 2, ..., \lfloor n/2 \rfloor$) in the Oval Index Triangle. The number of rhombs in the Oval for which the principal index is equal to q is equal to v_q, which is the qth component of the $\lfloor n/2 \rfloor$-dimensional *rhombic inventory vector* ('RIV').

RIV = [1 1 1 1 2 2 1 1 2 5 0 1 2 0 2 2 2 2 2 3 0 3].

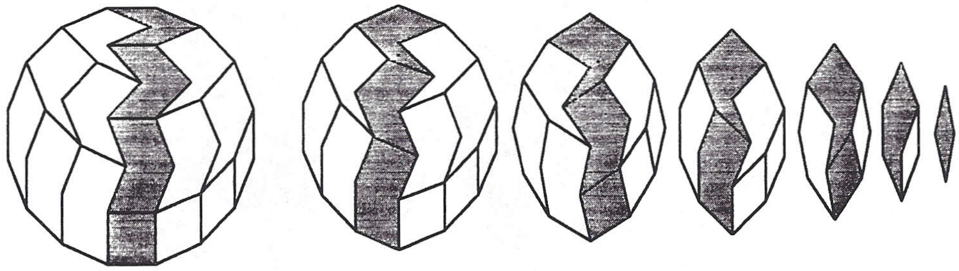
In a future edition of this manual, I will summarize solutions of the following intriguing problems concerning Ovals:

(i) For what values of n do there exist Ovals with RIV = [1 1 1 ... 1]?
(ii) For what values of n can the rhombs in SRI_{2n} be arranged to tile another convex polygon besides the regular $2n$-gon?
(iii) For what values of n can the rhombs contained in an integer number of SRI_{2n} sets be partitioned to tile an integer number of *congruent* Ovals?

3.2 Monochrome 4-Ovals

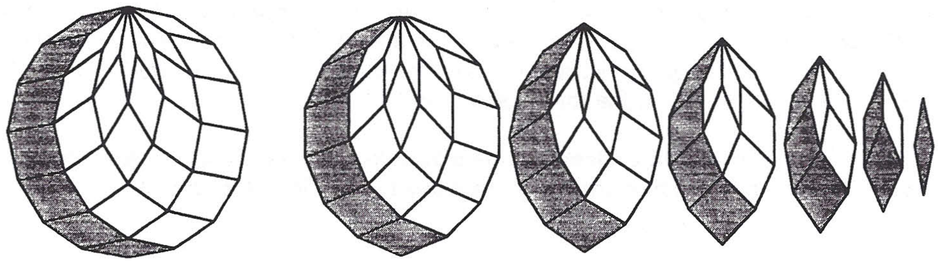
Use the three twin rombiks of any of the three graceful subsets of ROMBIX-16 to construct a 4-Oval—an Oval with eight sides. (For each graceful subset, there is only one such Oval.) For which of the four subsets is it possible to combine the three twin rombiks with the four keystones to construct a 5-Oval (ten sides)?

3.3 Hollowing Out the CRACKED EGG (tiling by *rombiks*)



Remove *n* ribbons from the CRACKED EGG, one by one, beginning with the central ribbon, to obtain a sequence of *n* Ovals of successively smaller sizes. Each ribbon removed is the one that contains the largest number of rhombs. After a ribbon is removed, form a smaller Oval from the remaining rombiks. For the first n-3 ribbons removed, this requires that the tiled fragment on one side of the removed ribbon be joined to the tiled fragment on the other side by a translation. The *digon* and the *point*, which are degenerate Ovals, are the last two Ovals in the sequence. (They are not shown in the sequence above.)

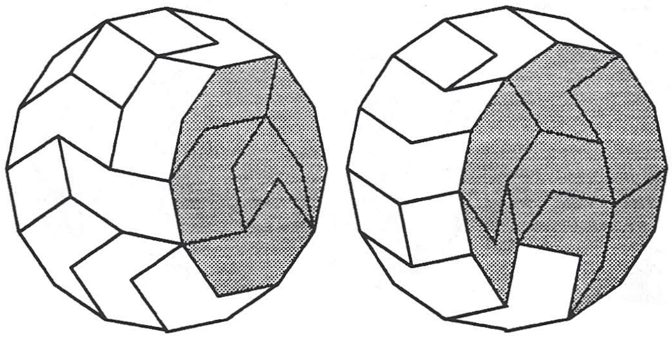
3.4 Shaving the STRAWBERRY (tiling by *rhombs*)



Peel off *n* strips of rhombs from the STRAWBERRY, one after another, to obtain a sequence of *n* Ovals of successively smaller sizes. In the *i*th strip ($i = 1, 2, ..., n$), there are n-i rhombs, leaving a g-Oval for which $g = n-i$. The last two Ovals in the sequence are the digon and the point (*cf.* §3.3).

3.5 Ovals Embedded in Circle Tilings by rombiks

If every edge of an Oval in a Circle Tiling by rombiks coincides with an *external* (boundary) edge of a rombik, we classify the Oval as *completely embedded*. If one or more edges of the Oval coincides with an *internal* edge of a twin rombik, *i.e.*, an edge common to the two rhombs that have been joined to make the twin, the Oval is called *incompletely embedded*. The Oval at lower left, which is fully shaded, is completely embedded; the partly shaded Oval at the right is incompletely embedded.



For at least some values of n, there exist Ovals that cannot be completely embedded in a Circle Tiling by rombiks. We conjecture, for example, that the incompletely embedded Oval for n=8 shown above cannot be realized as a completely embedded Oval.

We treat both (a) the rhombs of SRI_{2n} and (b) the $\{2n\}$ as Ovals (*cf.* the definition of an Oval in §3.1). As described in §3.3, we also classify as Ovals (c) the *point* and (d) the *digon*; the two edges of the digon are of the same length as a rhomb edge. The point is called the *null Oval*; the digon is called the *vacuous Oval*. We denote all four of these exceptional classes of Ovals as *trivial* Ovals.

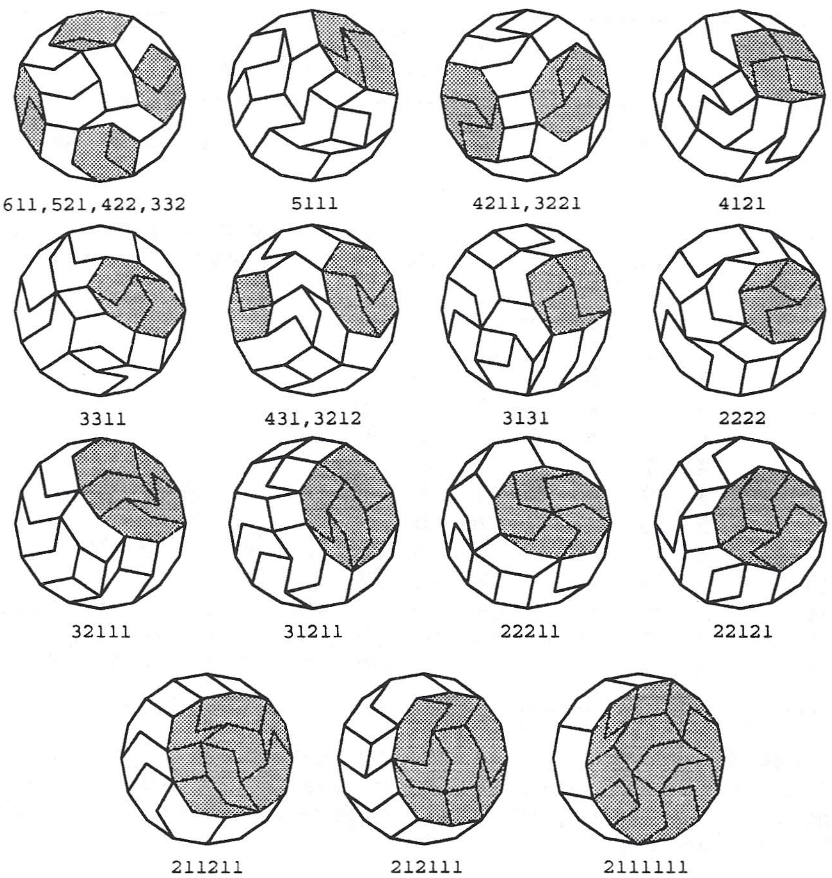
We conjecture that for every positive integer n, every non-trivial Oval can be realized as the boundary of a set of rhombs in a Circle Tiling by rhombs, and also that every non-trivial Oval can be embedded—either completely or incompletely—in a Circle Tiling by a proper subset of the rombiks in ROMBIX-$2n$.

The illustrations on p. 13 demonstrate that at least twenty of the twenty-three non-trivial Ovals for n=8 can be completely embedded in Circle Tilings for ROMBIX-16.

3.6 Pied 4-Ovals

Partition the twelve twin rombiks of ROMBIX-16 among four 4-Ovals, in each of which there are:

(a) two rombiks of one color and one of a second color;
(b) three rombiks of one color.



Twenty completely embedded ovals for n=8
Each shaded oval is identified by its Turning Angle Index Sequence (TAIS)

3.7 The association of Ovals in *conjugate pairs.*

Table 3.7.1 shows the number of g-Ovals for $1 \le n \le 16$. Inspection of the table reveals that for $n \le 16$, the number $M(n, g)$ of g-Ovals is exactly equal to the number $M(n, n\text{-}g)$ of $(n\text{-}g)$-Ovals. On closer inspection of the data for $n \le 13$, I discovered that if $g_1 + g_2 = n$ and $\sigma = |g_1 - g_2|/2$, then for every g_1-Oval O_1, there is a unique g_2-Oval O_2 with the same symmetry as O_1, and that the areas of O_1 and O_2 differ by exactly σ *kernels* (the *kernel* is defined here as the set of n-1 rhombs contained in a ladder). I call the Ovals O_1 and O_2 *conjugate*. The data also suggested that if n is a prime congruent to 3 (mod 4), then an integer number of sets of the rhombs in SRI_{2n} can be partitioned into congruent Ovals (in exactly two different ways). Alan McK Shorb and I subsequently proved these and a number of other properties of Ovals, using necklace theory and some theorems from elementary number theory. An account of this work will be published elsewhere.

Table 3.7.1

$M(n,g)$ values for $1 \leq n \leq 16$

g / n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
1	1	1	(2)															
2	1	1	1	(3)														
3	1	1	1	1	(4)													
4	1	1	2	1	1	(6)												
5	1	1	2	2	1	1	(8)											
6	1	1	3	3	3	1	1	(13)										
7	1	1	3	4	4	3	1	1	(18)									
8	1	1	4	5	8	5	4	1	1	(30)								
9	1	1	4	7	10	10	7	4	1	1	(46)							
10	1	1	5	8	16	16	16	8	5	1	1	(78)						
11	1	1	5	10	20	26	26	20	10	5	1	1	(126)					
12	1	1	6	12	29	38	50	38	29	12	6	1	1	(224)				
13	1	1	6	14	35	57	76	76	57	35	14	6	1	1	(360)			
14	1	1	7	16	47	79	126	133	126	79	47	16	7	1	1	(687)		
15	1	1	7	19	56	111	185	232	232	185	111	56	19	7	1	1	(1224)	
16	1	1	8	21	72	147	280	375	440	375	280	147	72	21	8	1	1	(2250)

The number in parentheses at the right of the row for Ovals of order n is the total number of g-Ovals of order n.

Data describing the conjugate relations for the thirty Ovals for n=8 are listed in Table 3.7.2. The thirty Ovals are numbered from -1 to 28. No. -1 is the Null Oval (point), and no. 0 is the Vacuous Oval (digon). We adopt the convention that the TAIS for the Null Oval is written as '[]'.

Curiously, six of the eight 4-Ovals are self-conjugate, but 4-Oval no. 12 is conjugate to 4-Oval no. 13. The algorithm that defines conjugacy and thereby specifies which Oval is conjugate to a given Oval is described in the forthcoming article written jointly with Alan Shorb. (It is based on the representation of the TAIS for the two conjugate Ovals as complementary sequences of integers on a 'circle diagram', or necklace.)

Table 3.7.2
The Conjugate Pairs of Ovals for $n = 8$
$u = (2\ 2\ 2\ 1)$ (RIV of kernel)

Oval no.	TAIS of Oval	RIV of Oval	TAIS of Conjugate Oval	RIV of Conjugate Oval	No. of Conjugate Oval	Symmetry group
g=0						
-1	[ ]	(0 0 0 0) + 4u	[11111111]	(8 8 8 4)	28	d16
g=1						
0	[8]	(0 0 0 0) + 3u	[2111111]	(6 6 6 3)	27	d2
g=2						
1	[71]	(1 0 0 0) + 2u	[311111]	(5 4 4 2)	23	d2
2	[62]	(0 1 0 0) + 2u	[221111]	(4 5 4 2)	24	d2
3	[53]	(0 0 1 0) + 2u	[212111]	(4 4 5 2)	25	d2
4	[44]	(0 0 0 1) + 2u	[211211]	(4 4 4 3)	26	d4
g=3						
5	[611]	(2 1 0 0) + u	[41111]	(4 3 2 1)	18	d2
6	[521]	(1 1 1 0) + u	[32111]	(3 3 3 1)	19	c2
7	[431]	(1 0 1 1) + u	[31121]	(3 2 3 2)	20	c2
8	[422]	(0 2 0 1) + u	[22211]	(2 4 2 2)	21	d2
9	[332]	(0 1 2 0) + u	[22121]	(2 3 4 1)	22	d2
g=4						
10	[5111]	(3 2 1 0)	[5111]	(3 2 1 0)	10	d2
11	[4211]	(2 2 1 1)	[4211]	(2 2 1 1)	11	c2
12	[4121]	(2 1 2 1)	[3311]	(2 1 2 1)	13	d2
13	[3311]	(2 1 2 1)	[4121]	(2 1 2 1)	12	d2
14	[3221]	(1 2 2 1)	[3221]	(1 2 2 1)	14	c2
15	[3212]	(1 2 3 0)	[3212]	(1 2 3 0)	15	d2
16	[3131]	(2 0 2 2)	[3131]	(2 0 2 2)	16	d4
17	[2222]	(0 4 0 2)	[2222]	(0 4 0 2)	17	d8

g=5

18	[41111]	(4 3 2 1) - u	[611]	(2 1 0 0)	5	d2
19	[32111]	(3 3 3 1) - u	[521]	(1 1 1 0)	6	c2
20	[31121]	(3 2 3 2) - u	[431]	(1 0 1 1)	7	c2
21	[22211]	(2 4 2 2) - u	[422]	(0 2 0 1)	8	d2
22	[22121]	(2 3 4 1) - u	[332]	(0 1 2 0)	9	d2

g=6

23	[311111]	(5 4 4 2) - 2u	[71]	(1 0 0 0)	1	d2
24	[221111]	(4 5 4 2) - 2u	[62]	(0 1 0 0)	2	d2
25	[212111]	(4 4 5 2) - 2u	[53]	(0 0 1 0)	3	d2
26	[211211]	(4 4 4 3) - 2u	[44]	(0 0 0 1)	4	d4

g=7

27	[2111111	(6 6 6 3) - 3u	[8]	(0 0 0 0)	0	d2

g=8

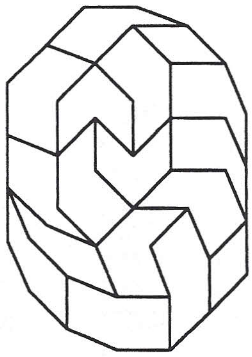
28	[11111111]	(8 8 8 4) - 4u	[ ]	(0 0 0 0)	-1	d16

3.8 Strictly Convex Ovals and Stretched Ovals

We call a *g*-Oval *Strictly Convex* if its boundary edges are all of the same length. If we relax this requirement, we obtain a class of polygons called *Stretched Ovals*.

DEFINITION 3.8.1
A Stretched Oval is a centro-symmetric convex polygon, each of whose turning angles is an integer multiple of π/n (integer $n \geq 2$), and each of whose edges is of length equal to some integer multiple of its smallest edge length.

The reader should be able to prove easily that every Stretched Oval can be tiled by replicas of the rhombs in SRI_{2n}. There is no limit to the number of rhombs contained in a Stretched Oval. The Stretched Oval at the right, which is tiled by rombiks drawn from two sets of ROMBIX-16, contains 32 rhombs; SRI_{16} contains 28 rhombs. Even if a Stretched Oval contains fewer rhombs than the number in SRI_{2n}, it may require more specimens of a given shape of rhomb than are contained in SRI_{2n}. In some cases, a Stretched Oval that can be tiled by the rhombs contained in SRI_{2n} cannot be tiled by the rombiks of a single ROMBIX set. **(Construct an example.)**



Since the only sets of parallel edges in a Circle Tiling by rhombs are rungs belonging to the same ladder, no edges incident on the same vertex are collinear. Consequently it is impossible for a Stretched Oval to be embedded in a Circle Tiling by rhombs.

The rombiks of ROMBIX-16 can be partitioned among 3, 4, 5, 6, 7, or 8 Strictly Convex Ovals, but a partition into two Strictly Convex Ovals is impossible, because 28 cannot be partitioned into two triangular numbers. There is, however, an elegant way to partition the rombiks of ROMBIX-16 into two *congruent Stretched Ovals.* It is even possible to segregate the four monochrome subsets in tilings of this pair of Stretched Ovals so that each of the Ovals is tiled by exactly two subsets whose common boundary is of the same shape in the two Ovals.

A computer search reveals that 7 of the first 15, 64 of the first 100, and 775 of the first 1000 triangular numbers can be partitioned into two triangular numbers.

What fraction of the first 10,000 triangular numbers can be partitioned into two (non-zero) triangular numbers? Can all but a finite number of triangular numbers be partitioned into two triangular numbers?

THEOREM 3.8.1
If *S* is a Strictly Convex Oval and is tiled by *p* rhombs of SRI_{2n}, then *p* is a triangular number $t(r) = 0, 1, 3, 6, ..., r(r+1)/2, ...$ ($r=0, 1, 2, ...$).
If we denote the number of edges of *S* by $2g(r)$, then $g(r) = r+1$.

As a consequence of Theorem 3.8.1, a partition of SRI_{2n} or of ROMBIX-$2n$ into a set of Ovals implies the representation of a triangular number as the sum of a set of smaller triangular numbers. This suggests the

---

CONJECTURE 3.8.1
Let $t(r)$ be any triangular number ≥ 10. Then for every $k \varepsilon [3, r+1]$, $t(r)$ can be expressed as the sum of *k* non-zero triangular numbers, which are not necessarily distinct.

Table 3.8.1 shows examples of partitions of the triangular numbers 10, 15, 21, 28, 36, and 45.

If Conjecture 3.8.1 is true, then for $n>4$ the rhombs in SRI_{2n} can be partitioned into k Strictly Convex Ovals tiled by rhombs, for every $k \varepsilon [3, n]$ (since $n=r+1$). I have found that for some values of n, there are too few keystones to allow partitions of ROMBIX-$2n$ into n or n-1 Ovals, but I have not found cases in which the partitioning of ROMBIX-$2n$ into more than three (but fewer than n-1) Ovals is impossible.

It is a well-known theorem [Beiler 1966] that every integer is either a triangular number, the sum of two triangular numbers, or the sum of three triangular numbers. However, I know of no references to Conjecture 3.8.1.

Table 3.8.1

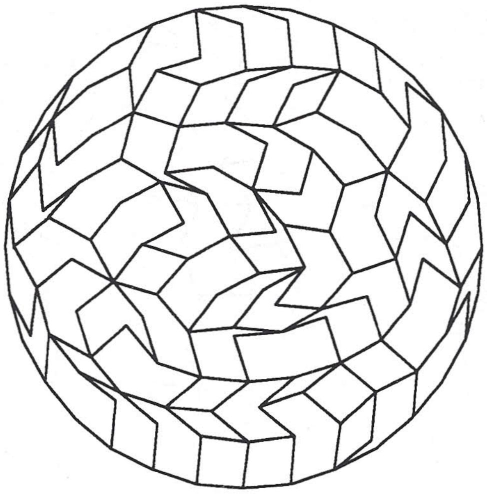
Examples of partitions of triangular numbers $t(r) = r(r+1)/2$ into k smaller triangular numbers ($3 \le k \le r+1$) for $4 \le r \le 9$

r	$t(r)$	$k = 3$	4	5	6	7	8	9	10
		Partition							
4	10	1+3+6	1+3+3+3	1+1+1+1+6					
5	15	3+6+6	1+1+3+10	1+1+1+6+6	1+1+1+3+3+6				
6	21	1+10+10	3+6+6+6	3+3+3+6+6	1+1+1+6+6+6	1+1+1+3+3+6+6			
7	28	3+10+15	6+6+6+10	3+3+6+6+10	1+1+3+3+10+10	1+3+3+3+6+6+6	1+1+1+1+6+6+6+6		
8	36	6+15+15	6+10+10+10	3+3+10+10+10	1+1+3+6+10+15	1+1+3+3+3+10+15	1+3+3+3+6+10+10	1+1+1+1+3+3+6+10+10	
9	45	15+15+15	10+10+10+15	1+3+10+10+21	1+3+6+10+10+15	3+3+3+6+10+10+10	1+1+1+6+6+10+10+10	1+3+3+3+3+6+6+10+10	3+3+3+3+3+6+6+6+6+6

4. MISCELLANY

4.1 Chaotic Circle Tilings for ROMBIX-2n sets for n>>8

Chaotic Circle Tilings by the rombiks of ROMBIX-2n have been found by the author for all $n \leq 16$. Shown below is an example of a chaotic Circle Tiling for n=16.

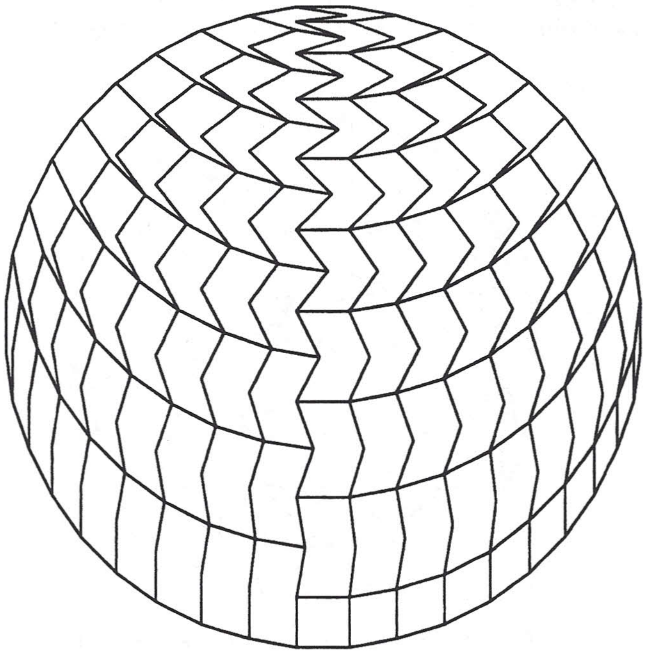


Chaotic Circle Tiling for n=16 (64 rombiks)

4.2 A Macintosh program for drawing the CRACKED EGG

Finding a chaotic Circle Tiling for a ROMBIX set of order n>12 or so is likely to be time-consuming. Whatever your preference, if you wish to make a set of rombiks of order $n \neq 8$, you may order a floppy containing a True BASIC program for the Macintosh called 'Cracked Egg Pattern' that prints the outlines of the rombiks in the CRACKED EGG for any ROMBIX set of order $n \leq$100,000. A listing of the program is included with the floppy. The program prints a drawing of the CRACKED EGG. You can freely select the size of the output image, a sample of which is shown on p. 20. A True BASIC Run Time program is also installed on the floppy. Cracked Egg Pattern works under System 6.08, but not System 7. After February, 1995, when True BASIC will be upgraded to System 7, I will be able to send you a version of Cracked Egg Pattern for System 7. Information about how to order this program is provided at the end of this manual.

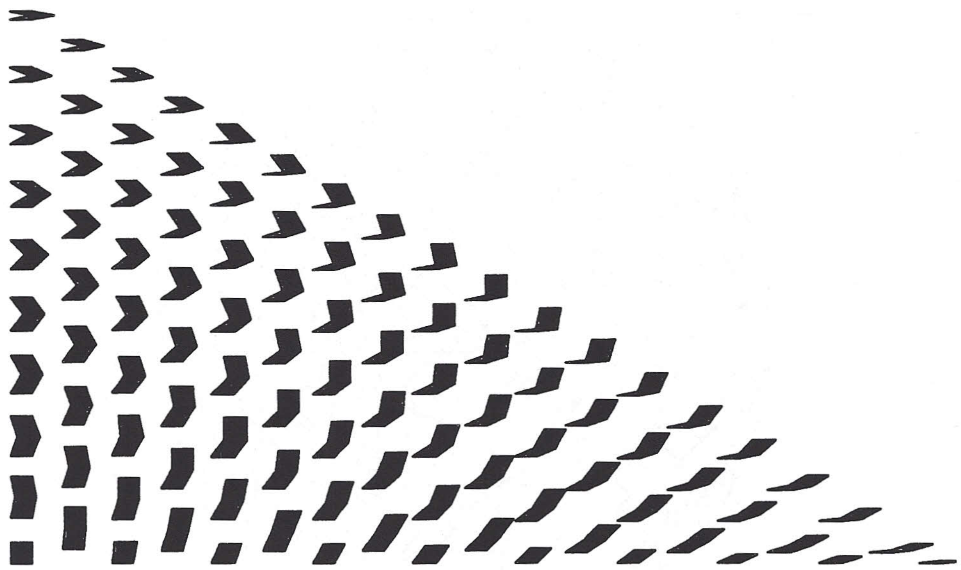
I disclaim responsibility for whatever mental anguish may result from your attempt to complete a chaotic ROMBIX Circle Tiling for large n.



The CRACKED EGG for n=20

4.3 The *Triangle Array* for ROMBIX-40

On p. 21 is shown an example for n=20 of a triangular arrangement of the rombiks of ROMBIX-2n that can be obtained by a simple transformation applied to the columns ('ribbons') of rombiks in the CRACKED EGG.



An orderly triangular arrangement of the 100 rombiks of ROMBIX-40

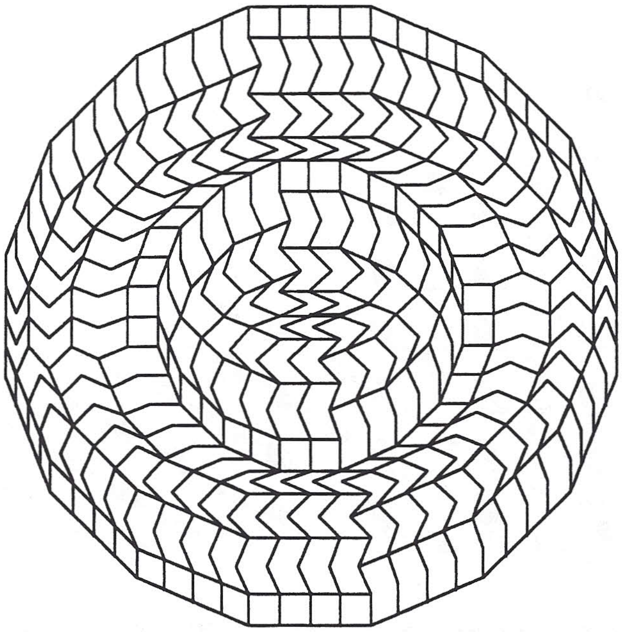
4.4 *Concentric* Nested Ring tilings by ROMBIX-2*n* sets

The illustration on the cover shows an orderly Concentric Nested Ring tiling for ROMBIX-40, a single set of which contains 100 rombiks. In the center of this pattern is a single set arranged in the CRACKED EGG Circle Tiling. This central core is surrounded by rings containing 8, 16, and 24 sets, respectively.

In an alternative construction, the core contains four sets arranged in the *Expanded CRACKED EGG* pattern, which is shown on p. 22. The core of this pattern requires rings containing 12, 20, 28, ... sets, respectively.

Orderly Concentric Nested Ring tilings are defined for ROMBIX sets of all orders. Chaotic Concentric Nested Ring tilings have also been found for the first few rings, for several values of *n*. It is conjectured that such chaotic tilings are possible for rings of all orders, for any ROMBIX set.

A four-set large Circle Tiling with c4 symmetry, which was discovered by Bill and Ruth Perk, is composed of four congruent simply-connected sectors, each tiled by one ROMBIX set. A sector can be derived from the Triangle Array (*cf.* §4.3) by reversing the left-right orientation of alternate columns in the Array and then collapsing the columns horizontally into a compact assembly.



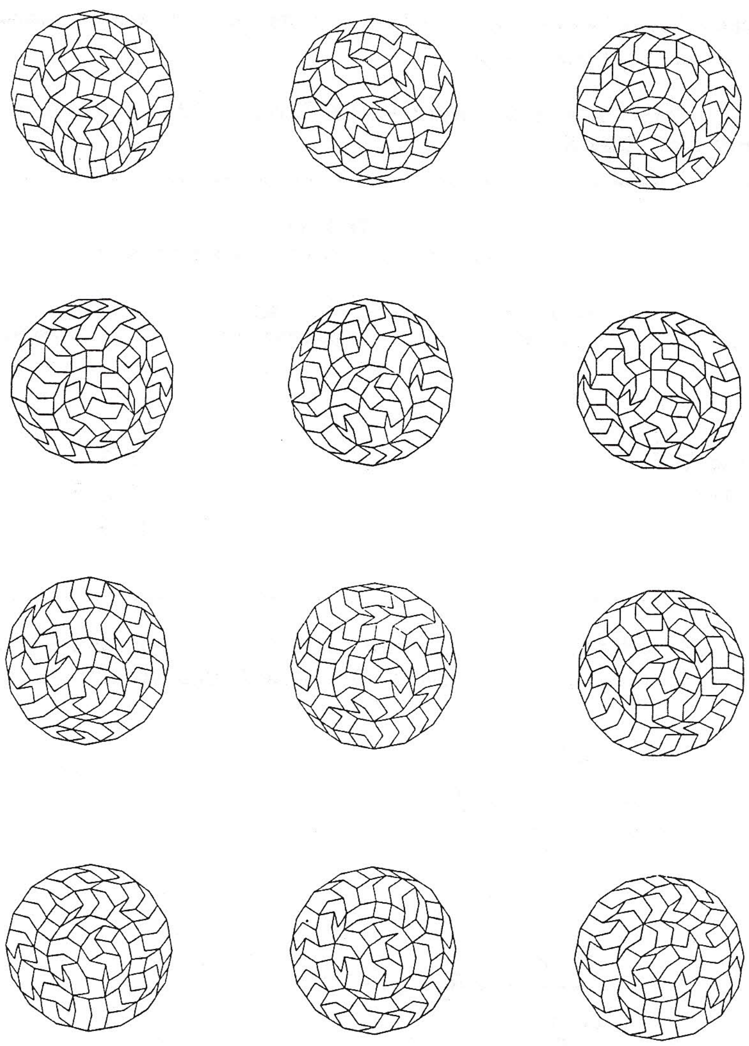
ORDERLY CONCENTRIC RING PATTERN FOR ROMBIX-16
THE CORE OF THE PATTERN IS THE *EXPANDED CRACKED EGG* (FOUR SETS)

4.5 Chaotic Eccentric Nested Ring tilings by ROMBIX-16 sets

Examples of Eccentric Nested Ring tilings for ROMBIX-16 are shown on p. 23 for the first annular ring surrounding a single-set core, for all twelve allowed values of the *eccentricity*, which is defined on p. 24. It appears that there exists no general recipe, analogous to the CRACKED EGG algorithm, for tiling orderly Eccentric Nested Rings.

There is one value of eccentricity for n=8 for which I have not yet found a tiling solution for the 3-set ring. If you carefully examine the illustrations of the twelve 3-set rings on p. 23, you will discover that in one of them, a certain twin rombik *T* appears twice, while its alternative isomeric form appears four times. I conjecture, however, that a valid tiling solution is possible. Please let me know if you find one!

I conjecture that there exist unbounded Eccentric Nested Ring tilings for every allowed value of eccentricity, for ROMBIX sets of all orders.



The eccentricity ε of an annular ring is defined as follows:

Let D = the distance between the centers of the two regular polygons that define the inner and outer boundaries, respectively, of the ring. Let R_{min} = the circumradius of the inner boundary polygon. Then

$\varepsilon = D / R_{min}$.

An equivalent definition is as follows: Let t_{max} = the projected radial length of the strip of rhombs that spans the ring sector of maximum radial width. Let t_{min} be defined similarly for the ring sector of minimum radial width. Then $t_{ladder} = t_{max} + t_{min}$, the projected radial length of a complete ladder, and

$\varepsilon = (t_{max} - t_{min}) / (t_{max} + t_{min})$.

For ROMBIX sets of any order, Nested Eccentric Ring patterns have the structure described in Table 4.5.1.

Table 4.5.1
Pattern structure of Nested Eccentric Rings

	Edge length m of outermost ring	No. of sets in outermost ring	Total no. of sets, including outermost ring and all inner rings
	1	1	(1 = 1 = 1^2)
SINGLE-SET	2	3	(1+3 = 4 = 2^2)
CORE	3	5	(1+3+5 = 9 = 3^2)
	4	7	(1+3+5+7 = 16 = 4^2)
	(etc.)		

The allowed values for the eccentricity are found by determining every set of *inventory coefficients* $\{a_1, a_2, ..., a_{n-1}\}$ (a_k=0 or 1) that satisfy both of the following inequalities:

$$\frac{\sum_{k=1}^{n-1} a_k \cos\frac{k\pi}{n}}{\sum_{k=1}^{n-1} |a_k| \sin\frac{k\pi}{n}} \geq \tan\frac{\pi}{2n}$$

$$\frac{1 - \sum_{k=1}^{n-1} a_k \cos\frac{k\pi}{n}}{\sum_{k=1}^{n-1} a_k \sin\frac{k\pi}{n}} \geq \tan\frac{\pi}{2n}$$

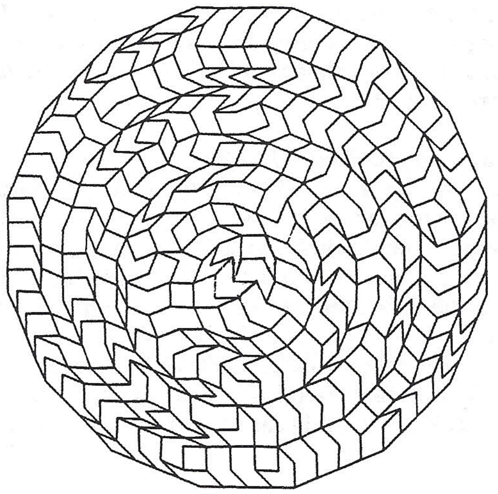
The value of the inventory coefficient for rhomb k (k=1, 2, ..., n-1) determines whether or not rhomb k is included in a ladder segment in the sector of minimum radial width in the ring: only if the inventory coefficient for rhomb k is equal to one is rhomb k included. Rhomb k is defined as a rhomb for which the angle between its bottom rung edge and its left edge is equal to $k\pi/n$. The existence of a solution set is a necessary condition for the existence of the corresponding Eccentric Nested Ring structure tiled by rhombs. I have been unable to prove, however, that the existence of a solution set is sufficient to guarantee that the Eccentric Nested Ring structure exists. Whether a Ring structure can be tiled by a suitable odd number of sets of *rombiks* can be determined only by trial and error.

Table 4.5.2 lists the twelve solution sets for rhombic inventory coefficients for n=8. These data were obtained from a computer program that generates all possible solution sets for any $n \leq 2$.

Table 4.5.2
The twelve Eccentric Nested Ring structures for *n*=8

	INVENTORY COEFFICIENTS						
ε	1	2	3	4	5	6	7
1.000000	0	0	0	0	0	0	0
~.844759	1	0	0	0	0	0	0
~.718695	0	1	0	0	0	0	0
~.632458	0	0	1	0	0	0	0
~.566454	1	0	0	0	0	1	0
~.480217	1	0	0	0	1	0	0
~.414214	1	1	0	0	0	0	1
~.351153	0	1	0	0	1	0	0
~.327976	1	0	1	0	0	0	1
~.234633	0	0	1	1	0	0	0
~.198912	1	0	1	0	0	1	0
~.082392	1	0	0	1	1	0	0

Shown below is a tiling of three Eccentric Nested Rings, surrounding a CRACKED EGG core, for ROMBIX-16.



4.6 Ten Tiling Tips

I offer here several suggestions that are likely to be found useful for finding chaotic Circle Tilings by ROMBIX sets for $n \geq 6$. The first two of these tips are not just suggestions—they are requirements. The other eight tips are rules derived from experience.

1. ***COLLINEAR EDGES***
   Never place one rombik ρ_1 next to another rombik ρ_2 in such a way that an edge of ρ_1 is collinear with an edge of ρ_2. This rule applies also to the *interior* edges of twin rombiks. (An interior edge is defined as an edge that is common to the two rhombic modules of a twin rombik.) The only parallel edges in a Circle Tiling are edges that are rungs of the same ladder.

2. ***LADDER COMPOSITION***
   Avoid placements of rombiks which violate the Ladder Composition Rule [Ball and Coxeter 1962]:

   In every Ladder there are n-1 rhombs.
   Each non-square rhomb occurs twice—once leaning to the left, and once leaning to the right.
   If n is even, the Ladder contains one square rhomb.

   When chaotic Circle Tilings for large n (e.g., n>>8) are attempted, it is found that unless the Ladder Composition Rule is consciously applied, it is likely to be violated. For smaller values of n, such violations are less likely to occur, once sufficient experience has been acquired so that faulty configurations can be recognized at a glance. However, if tip no. 4 in this list ('Work from one side to the other') is ignored, it becomes difficult to identify faulty configurations quickly. It is then necessary to check carefully for violations of the Ladder Composition Rule.

3. ***SAVE THE KEYSTONES***
   Try to postpone inserting the keystones until you have nearly finished the tiling. It is recommended that they be saved until most of the twins are already in place, since single rhombs will always fit into any legal hole. (This rule becomes steadily more important with increasing n.)

4. ***WORK FROM ONE SIDE TO THE OTHER***
   Begin the tiling by placing a rombik snugly against the boundary. As additional rombiks are added, try to place at least some of them in such a way that as many of their sides as possible coincide either with boundary edges or with sides of rombiks already placed. Don't allow the boundary between the placed rombiks and the empty part of the arena to become excessively long. On the other hand, if you maintain too smooth a boundary, you are likely to find it impossible to complete the tiling (except in the case of the CRACKED EGG). It is usually desirable to maintain a somewhat disorderly interface. Strive to create what the late Max Delbrück used to call *controlled chaos*. (Of course, it is easier to *recognize* controlled chaos than it is to *define* it!)

5. ***GET RID OF THE SKINNY CHEVRON FIRST***
The thinnest identical twin is in some respects the most awkward of all the rombiks. It is likely to resist finding a place in the Circle Tiling if you don't place it early.

6. ***SWAPPING (STRADDLE AND EXCHANGE)***
If all of the rombiks that have not yet been placed in the tiling are twins and don't fit anywhere, don't immediately give up and start over again. Instead, search for a place where an unplaced twin T_1 matches—and can therefore *straddle*—two sites in the tiling occupied by a pair of rhombs that belong to a pair of twins T_2 and T_3 with two common edges. After removing T_2 and T_3 and inserting T_1, you may succeed in rearranging the tiling in such a way that all of the rombiks eventually fall into place. Of course you will have to fill the hole you made when you removed T_2 and T_3 and inserted T_1!

7. ***ISOMER SWAPPING***
Every twin rombik that is neither an identical twin nor a twin that contains the square rhomb is one of a pair of *isomers*: the two isomers of every such pair are composed of the same two rhombs, but the rhombs are joined together along different edges so that they form two twins of different shape. Suppose that A_1 and A_2 are the two isomers of twin A, and B is another twin. Suppose also that there exists a configuration composed of four rhombs which can be tiled either by A_1 and B or by A_2 and B. If such a configuration—containing A_1, for example—appears in the tiling, but A_1 is needed elsewhere in the tiling, then if A_2 has not yet been placed, it can be exchanged for A_1.

8. ***PAIR SWAPPING***
It occasionally happens that a pair of twins A and B that share two edges also fit somewhere else in the arena, and that two other twins not yet placed fit in the space occupied by A and B and can therefore be exchanged for A and B.

9. ***OVAL TURNING***
This technique often facilitates swapping. Every Oval is symmetrical at least by rotation through a half-turn (c2 symmetry). Some Ovals have rotational symmetry c4, c6, *etc.*, and some Ovals are symmetrical by reflection in two or more lines—they have symmetry d2, d4, *etc.* Depending on its symmetry, an Oval can be rotated by a half-turn, or quarter-turn, *etc.* Ovals with reflection symmetry can be reflected in any of their lines of reflection. We call all such transformations *Oval turning*. Oval turning often makes it possible to place one or more rombiks in a strategically better place.

10. ***COMBINING KEYSTONES***
It is useful to place as many of the keystones as possible next to each other in an unfinished tiling, particularly with sets for which $n \gg 8$. A fraternal twin for which there is no matching hole in the tiling can always replace two adjoining keystones if they correspond to the rhombic modules of that twin and are suitably oriented with respect to each other. The two keystones then become available to occupy sites elsewhere in the tiling. Oval turning is sometimes an effective way to force the required migration of separated keystones into adjoining positions.

4.7 Time required for a chaotic Circle Tiling

When one tries to find chaotic Circle Tilings for sets larger than ROMBIX-16, it is discovered that it takes about twice as long to find a Circle Tiling for a set of order n+1 as it does for a set of order n. Let K_n denote the ratio T_{n+1}/T_n of average solution times for sets of order n and n+1. If the average solution time were proportional to the number of rombiks in the set, K_n would decrease with n, because the fractional increase in the number of rombiks per set decreases with n. The ratio $\lfloor (n+1)^2/4 \rfloor / \lfloor n^2/4 \rfloor$ of the number of rombiks in the set of order n+1 to the number in the set of order n is equal to $1 + \lfloor (n/2) \rfloor^{-1}$.

Experience suggests that K_n is nearly independent of n and that it has approximately the same value for different people, but it would require considerable effort to confirm this conjecture. Here we will simply assume that K_n is equal to 2 for every n. If we also assume that the average solution time for ROMBIX-32—which has 64 rombiks—is 12 hours, we obtain the rough estimates of average solution times for ROMBIX sets for $n \leq 20$ shown in Table 4.7.1. These estimated values are in reasonably good agreement with the author's recorded average solution times for Circle Tilings for $n \leq 16$.

Table 4.7.1
Predicted solution times for chaotic Circle Tilings by standard ROMBIX sets

n	days	hours	minutes	seconds
2				2
3				5
4				10
5				21
6				42
7			1	24
8			2	49
9			5	38
10			11	15
11			22	30
12			45	
13		1	30	
14		3		
15		6		
16		12		
17	1			
18	2			
19	4			
20	8			

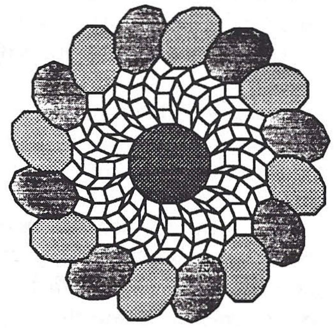
A factor which contributes to the large value of K_n is that the ratio of the number of keystones to the total number of rombiks in the set decreases with n. This ratio is equal to $\lfloor (n+1)/2 \rfloor^{-1}$. (This value suggests that K_n may be slightly smaller for even n than for odd n.) Keystones facilitate Circle Tilings, since—unlike twins—they can always be placed in any legal hole in the tiling. The relative paucity of keystones in sets of high order contibutes to the difficulty of Tiling the Circle. For $n > 10$ or so, skillful use of the keystones is crucial. It is desirable to hoard keystones as long as possible, and then use them to fill the holes remaining in the tiling (*cf.* Tiling Tip no. 3 on p. 26).

There is no theoretical upper limit to the order of a ROMBIX set, but because the solution time for Circle Tilings appears to increase exponentially with n, there is obviously a practical upper limit. The precise value of this limit is of course not well defined.

A purely physical constraint which places an upper limit on the order of ROMBIX sets that can be effectively made and handled is the size and shape of the smallest rhomb in SRI_{2n}. This limit is also not well defined, since it depends to some degree on both the thickness and the lengths of the sides of the rombiks, as well as on the material of which they are made.

In [Berlekamp, Conway, and Guy 1982 p. 790], Richard Guy discusses briefly some aspects of the complexity of a puzzle or search of "size" n, when the number of solutions varies as c^n. It would be of considerable interest to know the total number $T(n)$ of ROMBIX Circle Tilings (not counting two tilings that are related by rotation or reflection as distinct) for more than the first few values of n. All that is presently known is that for n=2, 3, 4, and 5, $T(n)$ = 1, 1, 3, and 15, respectively, and $T(6) >> 60$. If we define $R(n)$ as the total number of Circle Tilings by the rhombs of SRI_{2n}, then for n=2, 3, 4, and 5, we find that $R(n)$ = 1, 1, 1, and 6, respectively, and $R(6) > 60$.

4.8 Chipped Ovals and the tilings in which they are embedded.

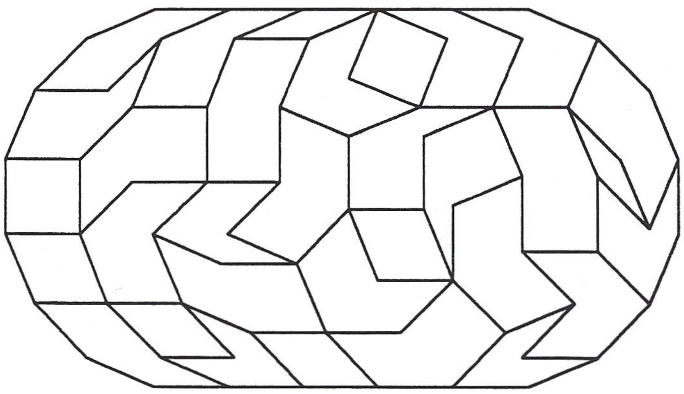


A Chipped Oval is a Strictly Convex Oval from whose boundary a simply-connected set of rhombs has been removed, leaving a concave indentation.

The *flower* tiling pattern grid shown at the right can be tiled by rombiks of ROMBIX-16. It was generated by first arranging sixteen congruent specimens of a Chipped Oval in a circular ring to form petals. Next the region in the interior of the ring was tiled from the outside in. Many variations on this theme are possible, since there exists a considerable variety of shapes of Chipped Ovals for ROMBIX-16. Either four, eight, or sixteen congruent specimens of such a Chipped Oval can be joined to form a ring of petals. If two different shapes of these Chipped Ovals are joined in an alternating pattern, a ring of thirty-two petals can be formed.

For some rings of petals, it is impossible to complete the flower tiling all the way to the center of the pattern. This situation is reminiscent of the 'essential holes' in Penrose patterns that were studied by J. H. Conway and described by Martin Gardner in his *Scientific American* article on Penrose patterns [Gardner 1977].

4.9 Tiling the two-set *RACETRACK* with segregated colors

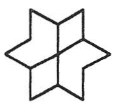


Use two ROMBIX-16 sets to tile the *Racetrack* (the Stretched Oval at the right) with as few *color clumps* as possible. (A color clump is a simply-connected region of one color that consists of either one rombik, or else of two or more rombiks, each of which shares at least one edge with another rombik.) The number of color clumps in the tiling illustrated above is eighteen, but I've managed to do it with six. Perhaps it's possible to do better than six.

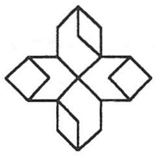
Can a two-set Racetrack for ROMBIX-16 be stretched horizontally into a three-set Racetrack? Four-set? What is the largest value of n for which ROMBIX-$2n$ sets can form Racetracks of unlimited length? Can you find an algorithm that defines a systematic tiling pattern for this largest value of n?

4.10 Stars of order *n* tiled by two sets of ROMBIX-2*n*

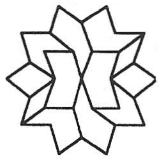
n = 2



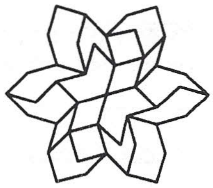
n = 3



n = 4

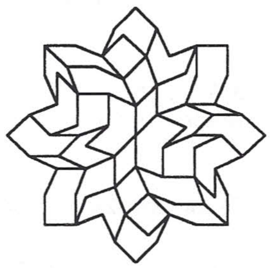


n = 5

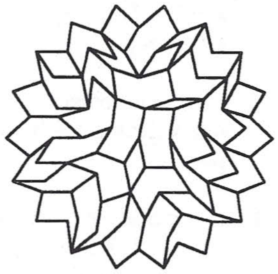


n = 6

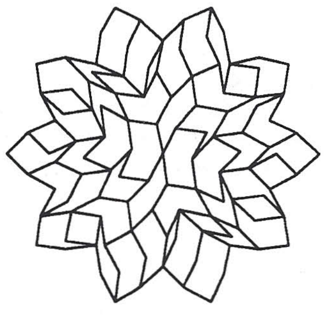
n=7



n=8



n=9



n=10

ROMBIX Stars
2≤*n*≤10

Examples of ROMBIX two-set tilings of Stars for all values of $n \leq 10$ are shown above. What is the smallest value of *n* for which it is impossible to tile the Star of order *n*? Can you construct an impossibility proof? Is it possible to use $2m^2$ sets ($m>1$) of rombiks to tile large Stars? Is it possible to use ROMBIX sets to tile Concentric Rings of Stars? For which values of *n* can Stars be tiled in patterns with c2 symmetry? d2 symmetry?

4.11 The tiling of Stars by the rhombs in two sets of SRI_{2n}

THEOREM 4.11.1
Every *n*-star can be tiled by the rhombs in two sets of SRI_{2n}.

Skecth of proof:

1. Construct a star-like central ring (tier 1) of $2n$ congruent rhombs, each with smaller face angle π/n (index=1), and each with a corner at the origin. If n=2 or n=3, go to step 4.
2. Construct a second ring (tier 2) of $2n$ congruent rhombs by inserting a rhomb with smaller face angle $2\pi/n$ (index=2) in each of the $2n$ notches of tier 1. If n=4 or n=5, go to step 4.
3. Construct consecutive rings k=3, 4, ..., $[n/2]$ of $2n$ congruent rhombs each by inserting a rhomb with smaller face angle $k\pi/n$ in each of the $2n$ notches of the previous tier.
4. If n is odd, the Star is complete. It contains $2n$ rhombs each for indices
   1, 2, ..., $(n-1)/2$.
   If n is even, the Star requires the removal of alternate rhombs (squares) from tier $n/2$. The completed Star contains $2n$ rhombs each for indices
   1, 2, ..., $n/2-1$,
   and n rhombs of species $n/2$.
   For both odd and even n, the Star comprises two sets of SRI_{2n}.

4.12 The shape of the outline of a Star

We have already seen that ROMBIX Stars have somewhat different shapes for odd and even n. The symbol for a regular star polygon is $\{p/q\}$; p is the number of edges and q is the *density* (or winding number) [Coxeter 1964]. The density is a measure of the separation between the two vertices of the regular *convex* polygon $\{p\}$ (= $\{p/q\}|_{q=1}$) which are incident on an edge of the star polygon.

DEFINITION 4.12.1

ODD *n*

For odd $n \geq 3$, a ROMBIX *star* (or *n-star*) is the interior of the regular star polygon $\left\{\frac{2n}{(n+1)/2}\right\}$. Each rhomb incident at one of the $2n$ vertices of the star polygon is called a *star-tip*; it has smaller face angle $\theta_n = (n-1)\pi/2n$. If the star is inscribed in the unit circle, the edge length e of the star-tip is $e = \frac{\sin(\pi/2n)}{\sin\left[\left(\frac{3n-1}{4n}\right)\pi\right]}$.

EVEN *n*

For all even $n \geq 2$, a ROMBIX *star* (or *n-star*) is derived from the region in the interior of the regular star polygon $\left\{\frac{2n}{n/2}\right\}$ by the removal of *n* alternate star-tips. A star-tip is the square rhomb that is incident at each of the $2n$ vertices of the star polygon. If the star is inscribed in the unit circle, the edge length e of the star-tip is $e = \frac{\sin(\pi/2n)}{\sin\left[\left(\frac{3n-2}{4n}\right)\pi\right]}$.

4.13 Embedded hexagons—an open problem

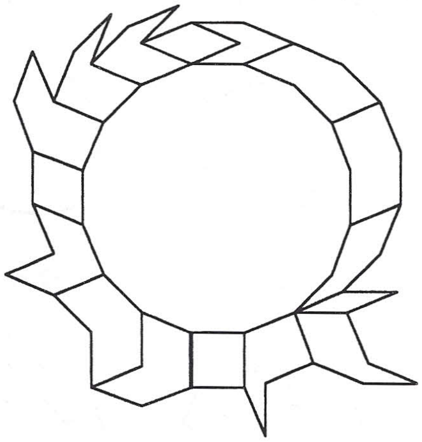
Make a chaotic Circle Tiling with the rombiks of ROMBIX-16. Count the number of embedded hexagons ('3-Ovals'). I conjecture that there are at least six 3-Ovals embedded in your tiling.

Can you prove (or disprove) the following *general* conjecture?

CONJECTURE 4.13.1
For $n \geq 2$, there are at least n-2 embedded 3-Ovals in every Circle Tiling by rhombs.

4.14 French's Fences (or 'the Farm problem')

Martin Gardner [Gardner 1986] describes four area maximizing problems for pentominoes, including one that was originally proposed by R. J. French [French 1939]. Victor G. Feser [Gardner 1986] independently resurrected this problem and added three related ones. French's problem is to construct a Farm of the largest possible area, subject to the condition that the fence be one unit wide.



I have translated French's original version of the Farm problem into the domain of ROMBIX as follows:

The Fence Problem for ROMBIX:
For the ROMBIX-$2n$ set, find the arrangement of pieces enclosing the largest possible area that can be tiled by rhombs from SRI_{2n}. The width of the fence must be nowhere less than the width of the thinnest keystone.

(What is the smallest value of *n* for which the area of the largest region that can be enclosed in a ROMBIX French's Fence is greater than or equal to the area of the corresponding regular $2n$-gon?)

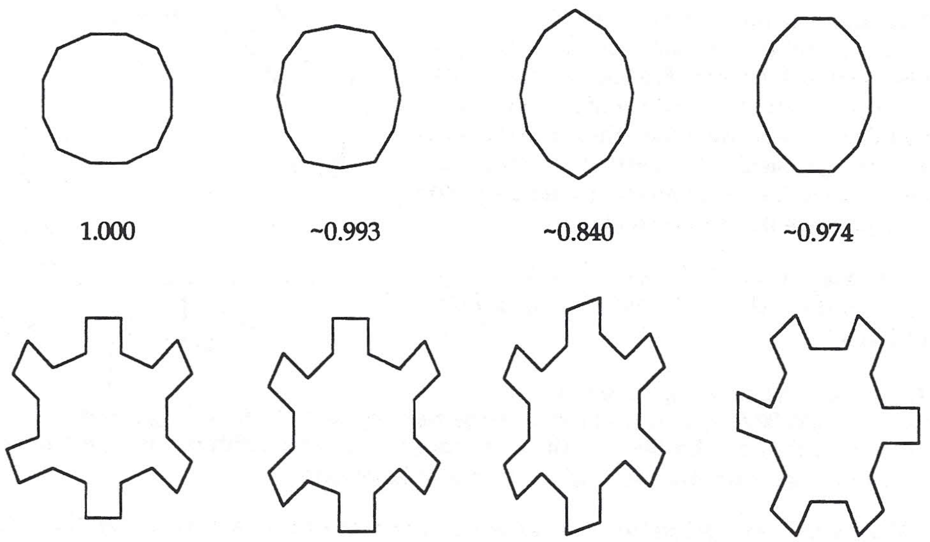
4.15 The two families of quasi-uniform tilings

On pp. 35-38 are shown examples of periodic tilings I call *quasi-uniform*. A quasi-uniform tiling is defined by two properties: (a) it is topologically equivalent to a uniform tiling [Grünbaum and Shephard 1987], and (b) all of its tiles are g-Ovals for n=8 (*cf.* §3.1), including at least one non-regular g-Oval. The tilings on pp. 35-36 are topologically equivalent to $(4 . 8^2)$; those on pp. 37-38 are topologically equivalent to (4 . 6 . 12). The only possible tiles in a quasi-uniform tiling are 6-Ovals, 4-Ovals, 3-Ovals, and 2-Ovals. The number of *aperiodic* quasi-uniform tilings in each of the two allowed families is infinite (as the reader can easily prove after examining the shapes of the outlines of the vertical columns of Ovals in certain of the illustrated tilings).

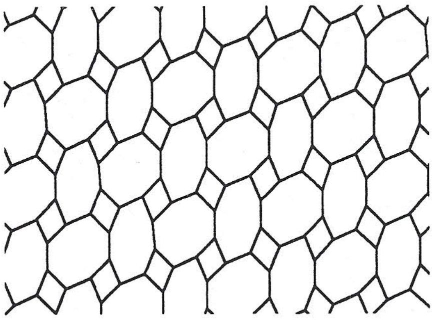
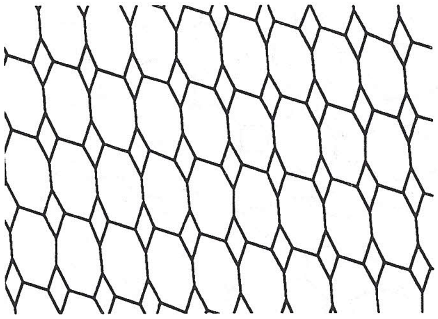
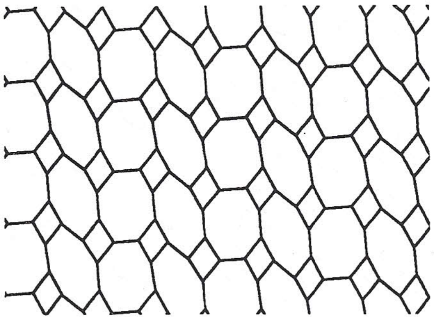
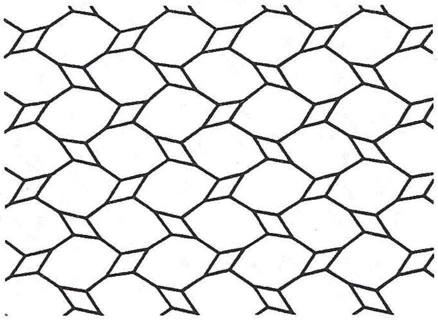
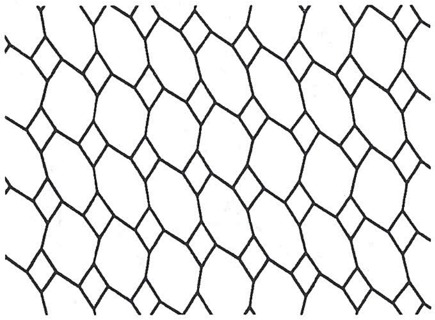
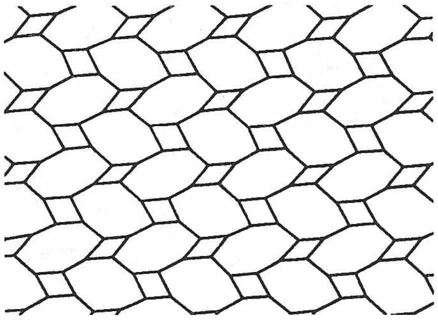
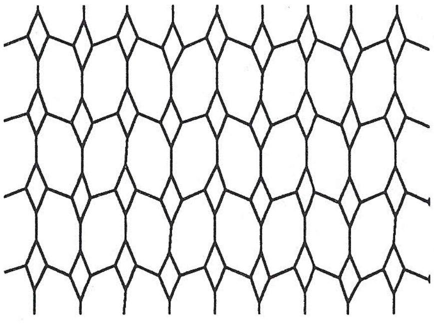
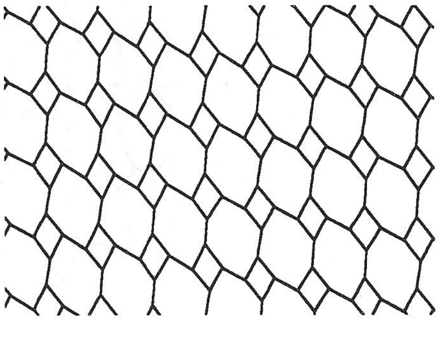
I have made a preliminary study of 'Turtle Glass', a hypothetical aperiodic pattern that is topologically equivalent to (4 . 6 . 12) but cannot be dissected into columns each of which has one-dimensional translation symmetry. There are altogether 1321 different configurations consisting of a 6-Oval whose twelve edges are incident on six 3-Ovals and six 2-Ovals in an alternating sequence, if:

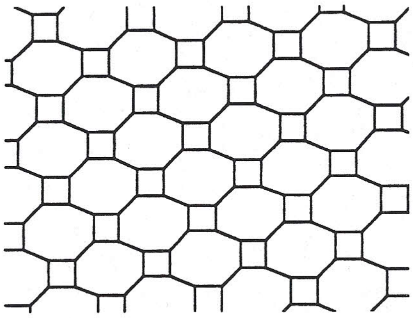
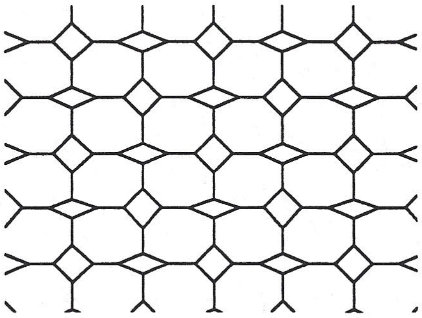
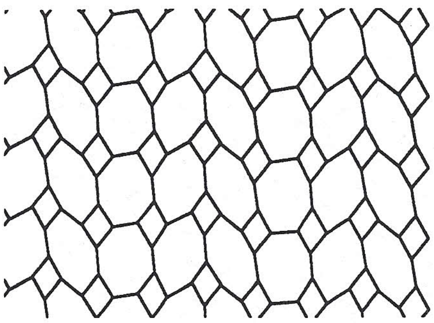
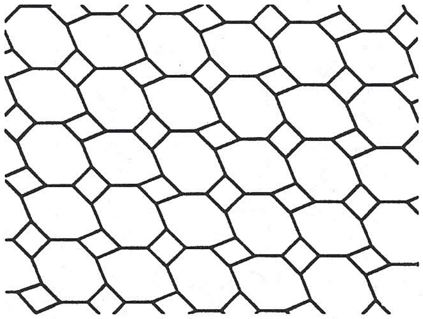
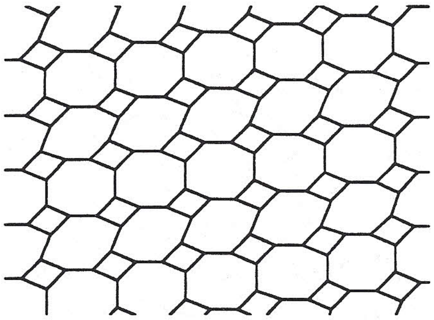
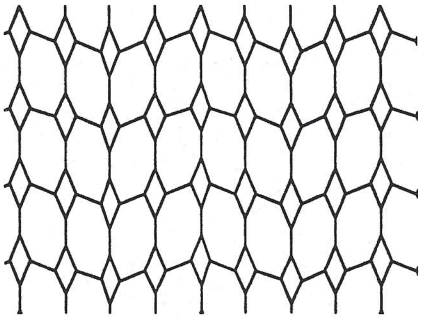
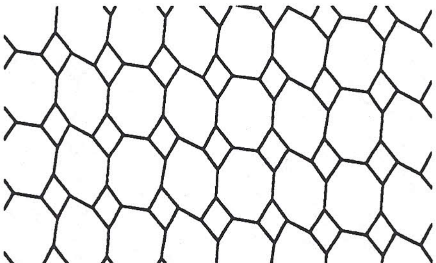
(a) each of the four shapes of 6-Oval for SRI_{16} is allowed, and
(b) the shapes of the 2-Ovals are limited to the nos. 3 and 4 rhombs (the 67.5° rhomb and the square), respectively.

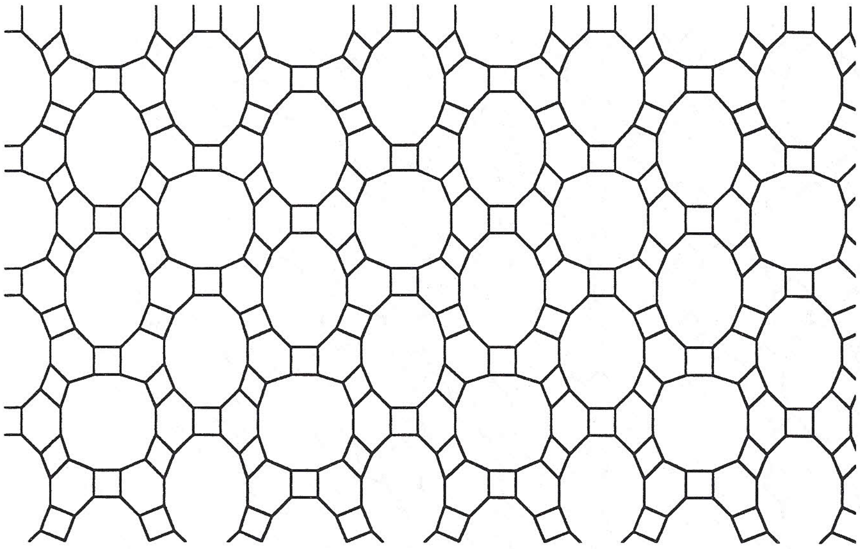
I have constructed a Turtle Glass pattern fragment (*cf.* p. 39) that contains a total of seventy-two 6-Ovals, including specimens of all four of the 6-Oval shapes. Twenty-three of the 1321 different configurations ('Turtles') of 6-Ovals and 2-Ovals appear in this pattern fragment, but I have no proof that the pattern can be extended indefinitely.

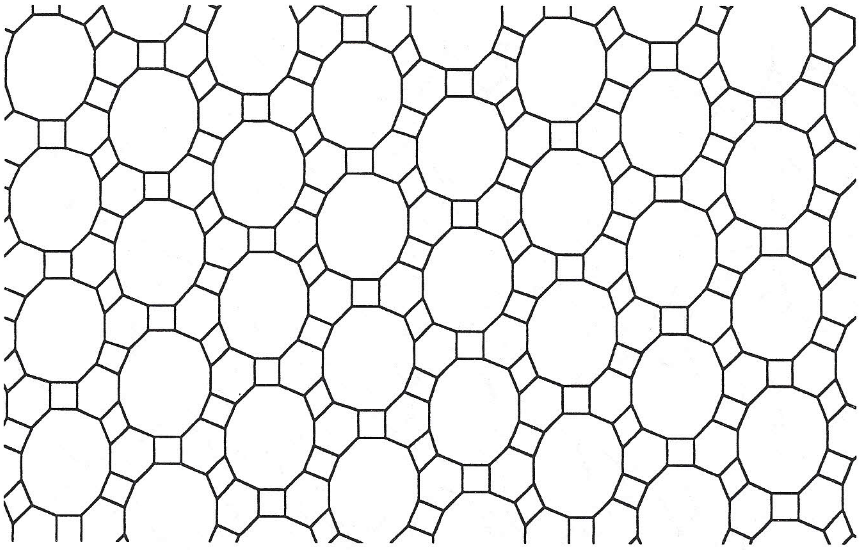


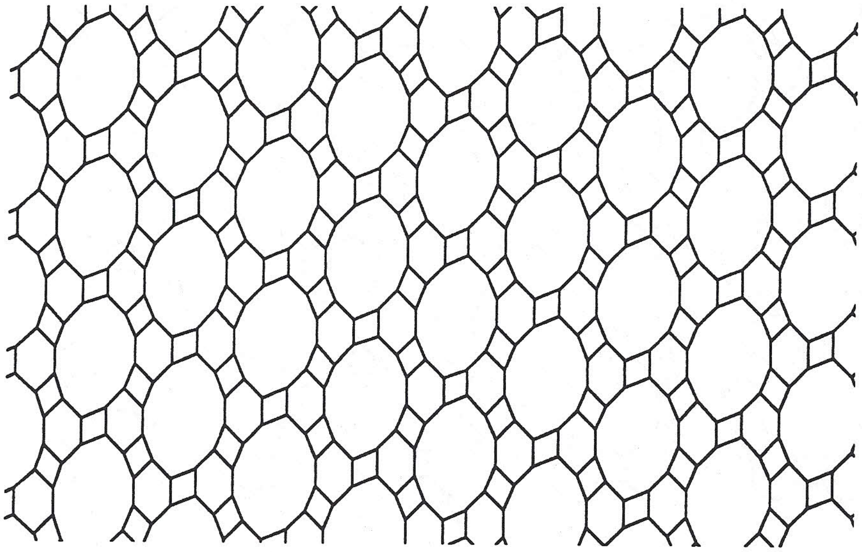
Shown above are the four shapes of 6-Ovals from SRI_{16}, together with their relative areas. Below each 6-Oval is shown a sample Turtle with the Oval as core.

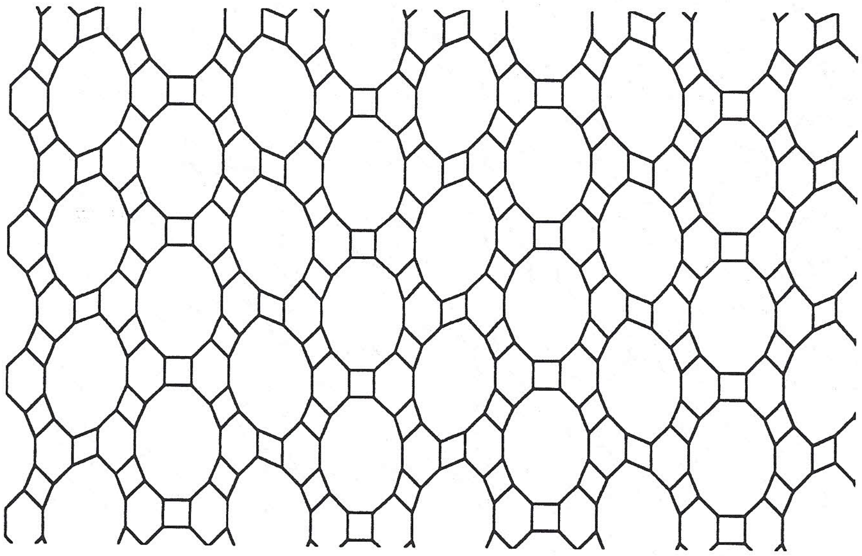









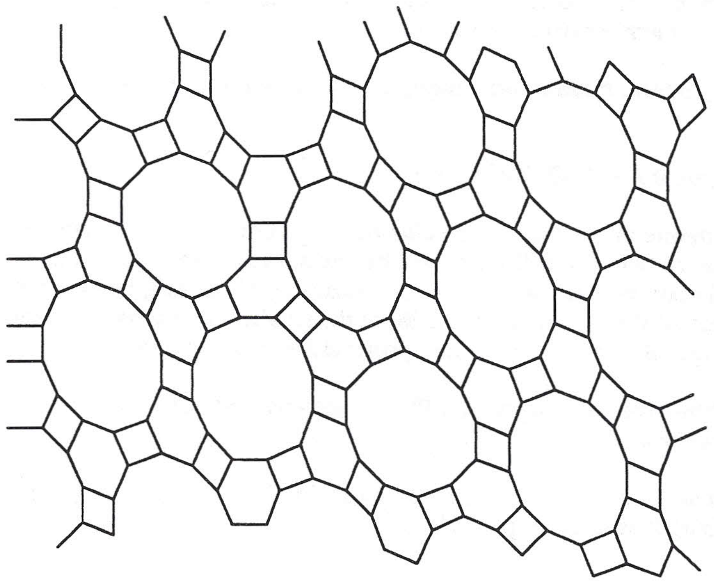








A small portion of a Turtle Glass pattern is shown below. Even if it were possible to prove that *infinite* chaotic Turtle Glass tilings exist, it is difficult to imagine that one could also prove that such a pattern can incorporate every one of the 1321 different Turtles. (For a discussion of a distantly related problem, *cf.* 'The Decidability of the Tiling Problem' [Grünbaum and Shephard 1987].)



4.16 Congruent Islands (ROMBIX-16)

1. Find a simply-connected region (*island*) I_4 that can be tiled by each of **four** disjoint sets of rombiks selected without regard to color from the ROMBIX-16 set.

2. Select **three** monochrome subsets of ROMBIX-16. Try to find an island I_3 that can be tiled by each of these three monochrome subsets of ROMBIX-16.

   a. For which three of the four possible selections of three subsets does I_3 exist?

   b. For the set of three subsets for which I_3 does *not* exist, find an island I_3* that can be tiled by each of three subsets, one of which is monochrome, the other two each containing three rombiks of one color and one twin rombik of a second color.

   c. Can you prove that no island can be tiled by all **four** subsets?

3. Partition the ROMBIX-16 set into two disjoint sets—A and B—of eight rombiks each. A and B are each comprised of two monochrome subsets. Discover the shape of an island I_2 that is *not* a Stretched Oval that can be tiled both by A and also by B.

4. Repeat task number 3 (immediately above), pairing the monochrome subsets in either of the other two possible ways.

4.17 Symmetrical islands (ROMBIX-16)

For each graceful subset, find an island that is not a ladder and has
(a) d1 symmetry or (b) c2 symmetry.

For each graceful subset, how many solutions are there of each symmetry type?

4.18 Congruent Atolls (ROMBIX-16)

Let us define an *atoll* as an annulus tiled by rombiks whose rhombic inventory is equal to that of two ROMBIX-16 monochrome subsets. We require that every point on the internal boundary of the atoll (*i.e.*, the boundary of the *hole*) be no closer to the external boundary of the atoll than the width of the smallest keystone. This width is defined as the orthogonal distance between opposite sides of the keystone.

1. Partition the rombiks of ROMBIX-16 into two sets, each of which comprises two monochrome subsets.

2. Find an atoll which can be tiled by each of these two sets. Make the total boundary length as small as possible.

4.19 LOOSE ENDS (ROMBIX-16)

The rules of this two-person game are summarized in the instructions in the ROMBIX-16 package. An experienced player will almost certainly place some of his rombiks in positions that create *holes* bordered either by rombiks or by both rombiks and one or more edges of the tray. If a hole accommodates one or more of a player's remaining rombiks but none of his opponent's, it is called a *haven*.

LOOSE ENDS is usually a short game, requiring at most only a few minutes of play. The game is too complex for complete analysis, but in any event it is desirable that the players take turns going first, in order to reduce any possible advantage to the second player. With practice, players learn that certain twin rombiks are somewhat more likely than others to fit into the arena near the end of the game. While it is important not to squander keystones early in the game, it is sometimes useful for a player to create a haven by placing a keystone in a hole even on an early move.

LOOSE ENDS is derived from a two-person game that was originally described by Golomb [Golomb 1965] [Golomb 1994 pp. 8-9]. In Golomb's version, one set of pentominoes is sufficient to make the game fascinating, but the ROMBIX-16 version is found to be thoroughly unsatisfactory if only one set of sixteen rombiks is used.

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NAME INDEX

Ball, W. W. Rouse, 4, 9, 26
Beiler, Albert H., 17
Berlekamp, Elwyn R., ii, 29

Conway, John H., ii, 29, 30
Coxeter, H. S. M., ii, 4, 9, 26, 32

Delbrück, Max, 26

Feser, Victor G., 33
French, R. J., 33

Gardner, Martin, i, ii, 30, 33
Geller, Ira, ii
Golomb, Solomon W., i, ii, 40
Grünbaum, Branko, i, ii, 34, 39
Guy, Richard K., ii, 29

Klarner, David A., i

Martin, George E., i

Penrose, Roger, ii, 30
Perk, H. F. W., ii, 21
Perk, Ruth, 21

Redmond, Donald, ii
Reid, Michael, i

Schoen, Alan H., i, 4, 5, 6, 7
Shorb, Alan McK, i, ii, 13, 14
Shephard Geoffrey C., i, ii, 34, 39