Some combinatorial properties of heterosets of even order n

Alan H. Schoen

Technical Report
Dept. of Electrical Engineering
College of Engineering
Southern Illinois University
Carbondale, IL 62901

March, 1994

Some combinatorial properties of heterosets of even order n

Abstract

We consider some combinatorial properties of *heterosets of order n*, which are defined as follows:

> Let n be any even positive integer, and let i, j, and k be integers ε $[1, n\text{-}1]$ such that
>
> $i \neq j \neq k.$
>
> The *heteroset i of order n* is comprised of
>
> the single *monomino i* and
>
> the $n/2 - 1$ *dominoes jk* for which $j + k \equiv 2i \bmod (n\text{-}1)$.

We analyze

> the *cyclic* structure of special sequences composed of the $n/2$ elements of a heteroset,

and

> the *matching* behavior of two heterosets of order n. Two heterosets i and j of order n are defined as matching iff the $n/2$ elements of heteroset i can be arranged so that the n-1 indices in the heteroset form a sequence identical to some sequence formed from the $n/2$ elements of heteroset j.

Some combinatorial properties of heterosets of even order n

Let n be any even positive integer.

Definitions:

1.1 A *monomino* i is a 1 x 1 square on whose top face is inscribed an integer *index* i from the set $\{1, 2, ..., n\text{-}1\}$. The bottom face of the monomino is left blank.

1.2 A *domino* jk is a 1 x 2 rectangle ($j, k \ \varepsilon \ \{1, 2, ..., n\text{-}1\}$; $j{\neq}k$). On the top face of one of its two unit squares, the index j is inscribed. On the top face of the other unit square, the index k is inscribed. The bottom faces of both unit squares are left blank.

1.3 Let $p \bmod^* q = p \bmod q + p\,\delta(p,q)$, where $\delta(p,q)$ is the Kronecker delta.

Heteroset i of order n ($i = 1, 2, ..., n$-1) is comprised of the single monomino i and the following $n/2$-1 dominoes (*cf.* the equivalent definition of heteroset i in the Abstract):

i+1	mod* (n-1)	i+n-2	mod* (n-1)
i+2	mod* (n-1)	i+n-3	mod* (n-1)
...			
i+n/2-2	mod* (n-1)	i+n/2+1	mod* (n-1)
i+n/2-1	mod* (n-1)	i+n/2	mod* (n-1)

The seven heterosets for n=8 contain the following elements:

heteroset	chord number	1	2	3	4	5	6	7
monominoes	(0)	1	2	3	4	5	6	7
	(1)	2 7	3 1	4 2	5 3	6 4	7 5	1 6
dominoes	(2)	3 6	4 7	5 1	6 2	7 3	1 4	2 5
	(3)	4 5	5 6	6 7	7 1	1 2	2 3	3 4

Table 1.1
Composition of the seven heterosets for n=8

It is convenient to use a *single-heteroset circle diagram* to represent each heteroset. Place n-1 *marked points*, numbered from 1 to n-1 in counter-clockwise order, at uniform angular intervals on the boundary of a circle. Point i represents the monomino of heteroset i. The remaining n-2 marked points are connected in disjoint pairs by parallel chords that represent dominoes. Each of the two marked points connected by a chord corresponds to one of the indices of a domino. The chords are numbered in consecutive order from 1 to n/2-1, starting from the chord that is closest to point i. The monomino is defined as a degenerate chord and is called the zeroth chord. The chord numbers are shown in parentheses in the first column of Table 1.1.

Fig. 1.1 shows the seven circle diagrams for the heterosets for n=8. In each diagram, except for the monomino, the chords are labelled by chord number.

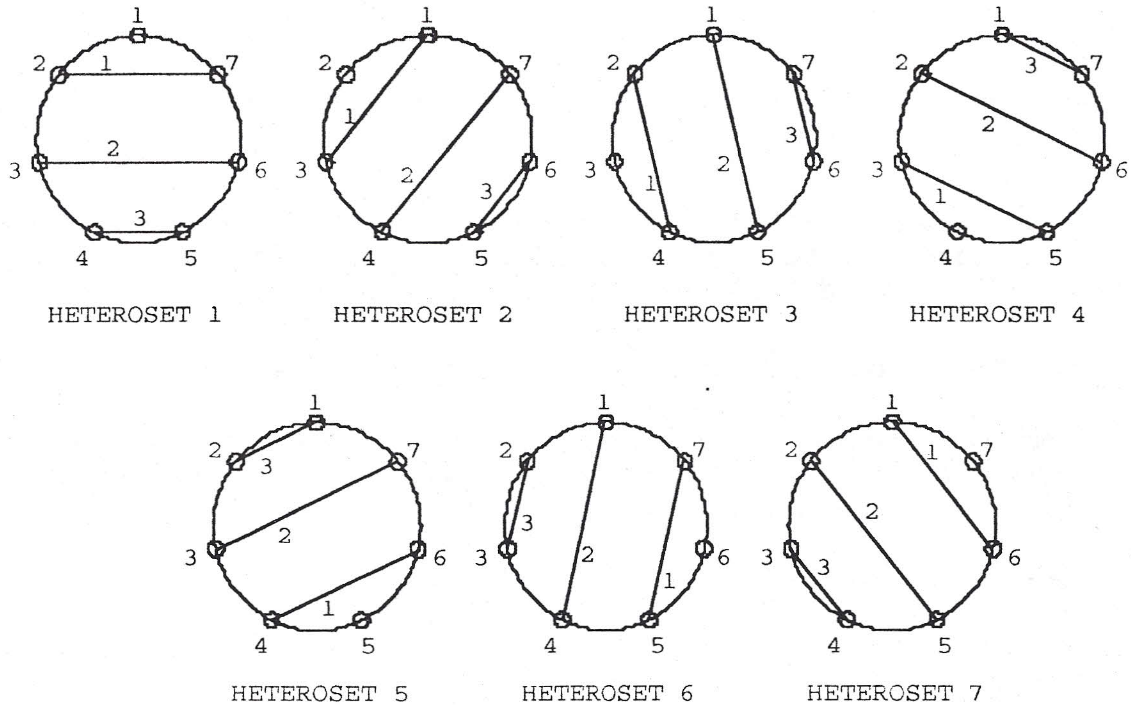


Fig. 1.1
Circle diagrams for the seven heterosets for *n*=8

We define the problem of Matching Heterosets as follows:

Define each domino ij in every heteroset as an unordered pair that has two possible *ordered states*: (ij) and (ji). Define the monomino i in the heteroset as ordered. Call any sequence composed of the $n/2$ elements of the heteroset, in their ordered states, an *index sequence.* If two heterosets can be ordered to form the same index sequence, they are called *matching.* Otherwise they are called *non-matching.*

For a given value of *n*, which pairs of distinct heterosets are matching?

Let us define the *cycle distance* $\Delta(j, k)$ between heterosets j and k as

$$\begin{aligned} \Delta(j, k) &= |k-j| && \text{if } |k-j| < n/2, \text{ and} \\ &= n-1 - |k-j| && \text{if } |k-j| \geq n/2. \end{aligned} \tag{1.1}$$

The cycle distance between heterosets j and k corresponds to the shorter of the two circular arcs that join j and k in the circle diagram. Because the indices of the elements in all of the heterosets are defined cyclically, whether or not two heterosets for given n are matching or non-matching depends only on the cycle distance between them. Let us call this property the Difference Rule.

It is proved in Theorem A.1 in the Appendix that a necessary condition for two heterosets to be matching is that their respective monominoes are located at opposite ends of the index sequence. Consequently it is impossible for three heterosets to be matching.

We can make use of a *two-heteroset circle diagram* to test for matching between two heterosets j and k. In this diagram, the chord sets of the single-heteroset circle diagrams for heterosets j and k are inscribed in the same circle. Heterosets j and k match iff there exists a connected path between marked points j and k, consisting of a sequence of chords selected alternately from the two heterosets, which includes every chord in each heteroset.

For n = 4, 6, and 8, every pair of heterosets is matching. Fig. 1.2 shows two-heteroset circle diagrams for n=8 for $\Delta(1, k)$ = 1, 2, and 3.

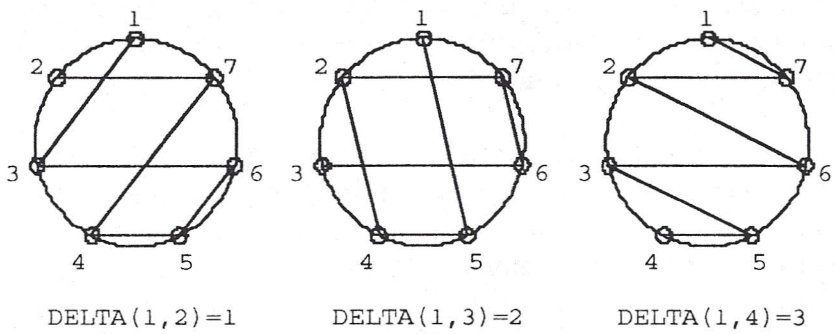


Fig. 1.2
Two-heteroset circle diagrams for n=8

For n=10, except for those heterosets for which $\Delta(1,k)$ = 3, *viz.*, 1:4, 2:5, 3:6, 4:7, 5:8, 6:9, 7:1, 8:2, and 9:3, every pair of heterosets is matching. Fig. 1.3 shows two-heteroset circle diagrams for n=10 for $\Delta(1, k)$ = 1, 2, 3, and 4.

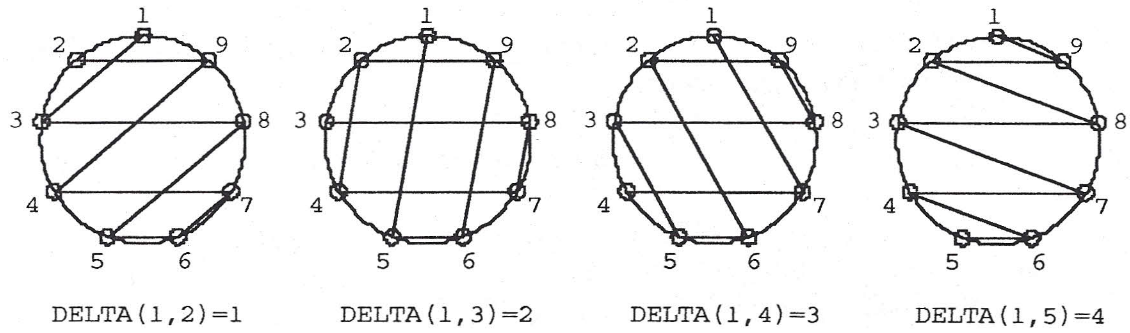


Fig. 1.3
Two-heteroset circle diagrams for n=10

For n=12 and n=14, every pair of heterosets is matching. For n=16, all pairs of heterosets for which Δ = 1, 2, 4, and 7 are matching; those for which Δ = 3, 5, or 6 are non-matching.

Definitions:

1.4 A heteroset i is called a *core heteroset* if $i \leq n/2$.

1.5 The indices i and n-i are called *conjugate indices.*

1.6 Two heterosets i and n-i are called *conjugate heterosets.*

1.7 The monomino i=n/2 is called *self-conjugate*.

1.8 The domino i n-i is called *self-conjugate*.

1.9 A heteroset i is called *balanced* if it forms a matching pair with its conjugate heteroset n-i, the respective monominoes i and n-i lying at opposite ends of the matching pair. Otherwise, it is called *unbalanced.*

Example: For n=8, core heterosets 1, 2, and 3 are balanced. The following arrangements define a matching pair for each of these heterosets and its conjugate:

heterosets 1 and 7	heterosets 2 and 6	heterosets 3 and 5
(1) (6 3) (4 5) (2 7)	(2) (3 1) (4 7) (5 6)	(3) (7 6) (4 2) (1 5)
(1 6) (3 4) (5 2) (7)	(2 3) (1 4) (7 5) (6)	(3 7) (6 4) (2 1) (5)

For n=10, heterosets 1, 3, and 4 are balanced, but heteroset 2 is unbalanced.

For every even n, heteroset n/2 is unbalanced. It is impossible to satisfy the requirement that the monomino n/2 and its conjugate—also equal to n/2—lie at opposite ends of the two index sequences that form a matching pair, because the index n/2 of the monomino occurs only once in heteroset n/2.

Theorem 1.1: If heterosets i and n/2 form a matching pair, then heteroset i is balanced.

Proof: Since heteroset i and heteroset n/2 form a matching pair, their elements can be arranged to form the following sequences:

heteroset	index sequence							
	(1)	(2)	(3)	(4)	(5)	(6) ...	(n/2-1)	(n/2)
i	(n/2 n-a)	(a n-b)	(b n-c)	(c n-d)	(d n-e)	(e ...) ...	(... n-i)	(i)
n/2	(n/2)	(n-a a)	(n-b b)	(n-c c)	(n-d d)	(n-e e) (...	) ... (n-i	i)

Now rearrange the elements of heteroset i as follows:

a. Reverse the order of the indices in each domino that is in an even element position (2), (4), ... Call such dominoes 'transposed' dominoes.

b. Translate each transposed domino to the left of the domino ($n/2$ n-a), which is in element position (1). The first transposed domino—the one that was in element position (2)—is moved into the position at the left of the domino ($n/2$ n-a). The second transposed domino—the one that was in element position (4)—is moved into the position at the left of the first transposed domino, *etc.*

The sequence produced by this rearrangement is shown below, demonstrating that heterosets i and n-i are matching:

heteroset — index sequence

(n/2-1) ... (6) (4) (2) (1) (3) (5) ... (n/2-2) (n/2)
i (n-i ...) ... (... e) (n-d c) (n-b a) (n/2 n-a) (b n-c) (d n-e) ... (...) (i)
n-i (n-i) (...) (e n-d) (c n-b) (a n/2) (n-a b) (n-c d) (n-e ...) ... (... i)
(n/2 odd)

or

heteroset — index sequence

(n/2) (n/2-2) (4) (2) (1) (3) (5) (n/2-1)
i (i) (...) ... (n-d c) (n-b a) (n/2 n-a) (n-c b) (n-e d) ... (... n-i)
n-i (i ...) (... n-d) (c n-b) (a n/2) (n-a n-c) (b n-e) (d ...) ... (n-i)
(n/2 even)

Using the inverse transformation, we can easily prove the converse of Theorem 1.1:

Theorem 1.2:

If heteroset i is balanced, then heterosets i and $n/2$ form a matching pair.

Example: Consider heteroset 6 for n=16, which matches heteroset 8 and is therefore balanced. The elements of heteroset 6 are

6, 7 5, 8 4, 9 3, 10 2, 11 1, 12 15, 13 14.

The elements of heteroset 10, which is conjugate to heteroset 6, are

10, 11 9, 12 8, 13 7, 14 6, 15 5, 1 4, 2 3.

The elements of heteroset 8 are

8, 9 7, 10 6, 11 5, 12 4, 13 3, 14 2, 15 1.

The arrangements which define heterosets 6 and 8 as matching are as follows (the eight element position numbers are shown for heteroset 6):

heteroset — index sequence

(1) (2) (3) (4) (5) (6) (7) (8)
6 (8 4) (12 15) (1 11) (5 7) (9 3) (13 14) (2 10) (6)
8 (8) (4 12) (15 1) (11 5) (7 9) (3 13) (14 2) (10 6)

The arrangements which define heterosets 6 and 10 as balanced are as follows:

heteroset	index sequence														
	(8)	(6)		(4)		(2)		(1)		(3)		(5)		(7)	
6	(6)	(14	13)	(7	5)	(15	12)	(8	4)	(1	11)	(9	3)	(2	10)
10	(6	14)	(13	7)	(5	15)	(12	8)	(4	1)	(11	9)	(3	2)	(10)

Definition:

1.10 A *string* is a sequential arrangement of the elements of a heteroset in which the self-conjugate index $n/2$ is located at one end, and the two indices a and n-a of every conjugate pair occur in index positions that are adjacent but are in distinct elements of the heteroset:

$$(n/2 \quad n\text{-}a)\ (a \quad n\text{-}b)\ (b \quad n\text{-}c)\ (c \quad n\text{-}d)\ (d \quad n\text{-}e)\ (e \quad ...)\ \ ...\ (... \ n\text{-}i)\ \ (i)$$

In the proof of Theorem 1.1, it was shown that if $j<n/2$, heteroset j forms a string iff heterosets j and $n/2$ are matching.

Heteroset $n/2$ cannot form a string, since the two indices of every conjugate pair occur together in single elements (dominoes) of the heteroset.

Theorem 1.3:

If heteroset i forms a string, the indices i and $n/2$ lie at opposite ends of the string.

Proof:

Let us denote by index position 1 the position at the end of the string that is occupied by the index $n/2$.

Next consider the index i, which is contained in a monomino. If it were in an interior index position, the indices in both of the index positions adjacent to it would be n-i. This is impossible, since each index occurs only once in a heteroset. Hence the index i occupies an end position in the string. Since the index position 1 is occupied by the index $n/2$, the index i occupies index position $n/2$.

Definitions:

1.11 A heteroset is called *unbroken* if it forms a string.
Otherwise, it is called *broken*.

1.12 A *stringlet* is a sequential arrangement of all of the elements—including the monomino i—of a proper subset of heteroset i, which contains the index $n/2$ in position 1. The index i is in position m, where m is equal to the total number of indices in the stringlet. Every pair of conjugate indices a and n-a occur in adjacent index positions—but in distinct elements—of the heteroset (just as in a string).

1.13 An *index cycle* is a cyclical arrangement of the p dominoes contained in a proper subset of the $n/2$-1 dominoes of the i-th heteroset. Every pair of conjugate indices a and n-a occur in adjacent index positions of neighboring elements of the heteroset.

Heterosets 1, 2, and 3 for n=8 are unbroken. The strings for these heterosets are as follows:

heteroset 1	heteroset 2	heteroset 3
(4 5) (3 6) (2 7) (1)	(4 7) (1 3) (5 6) (2)	(4 2) (6 7) (1 5) (3)

For n=10, heterosets 1, 3, and 4 are unbroken, but heteroset 2 is broken. If the index 5 is in index position 1, then the monomino 2 is forced into index position 3, thereby making it impossible to complete an entire string sequence: (5 8)(2)

Theorem 1.4: A string contains no index cycles.

Proof: Let S be the string for the unbroken heteroset i. Assume that the index cycle C, which is composed of the index pairs for p dominoes, is contained in S. Now express C as a linear strip of dominoes C', by cutting C along the boundary between the index pairs for any two adjacent dominoes D_1 and D_2. Denote the conjugate indices in the end positions of C' by a and n-a. It is impossible for C' to be contained in S, because the two conjugate indices a and n-a are not in adjacent index positions in S. (In a string, the two indices of every conjugate pair occur in adjacent positions.)

Theorem 1.5: Every domino in heteroset $n/2$ is an index cycle.

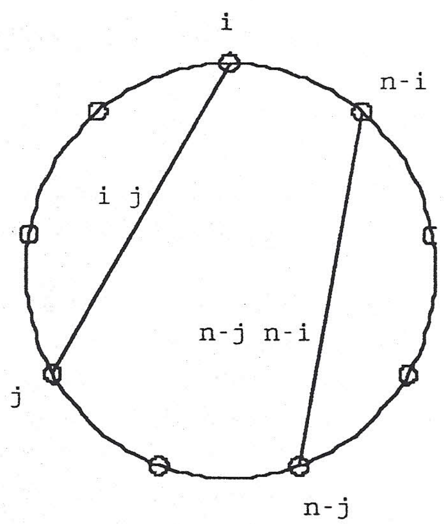
Proof: Every domino in heteroset $n/2$ is of the form $i\ n$-i, which is an index cycle.

Lemma 1.1: If—in the circle diagram for n—the chords $i\,j$ and n-j n-i ($i, j \le n$-1) are parallel, then either i=j or i=n-j. Hence the four points i, j, n-i, n-j are not all distinct.

Proof: The circle diagram for n contains the marked points 1, 2, ..., n-1. Consider marked points a and b on the diagram. Let $\Delta(a,b)$ be the cycle distance between a and b.

If $|b\text{-}a| < n/2$, $\Delta = |b\text{-}a|$;

if $|b\text{-}a| \ge n/2$, $\Delta = n\text{-}1 - |b\text{-}a| \pmod{n\text{-}1}$.



If the chords $i\,j$ and n-j n-i are parallel, then

$\Delta(n\text{-}i, i) = \Delta(n\text{-}j, j)$.

We will prove that for the four possible cases

(a) $n\text{-}2j < n/2$; $2i\text{-}n < n/2$,
(b) $n\text{-}2j \ge n/2$; $2i\text{-}n \ge n/2$,
(c) $n\text{-}2j < n/2$; $2i\text{-}n \ge n/2$,
(d) $n\text{-}2j \ge n/2$; $2i\text{-}n < n/2$,

the assumption that $\Delta(n\text{-}i, i) = \Delta(n\text{-}j, j)$ implies that the four points i, j, n-i, n-j are not all distinct.

(a) $\underline{n\text{-}2j < n/2;\ 2i\text{-}n < n/2}$

$[\Delta(n\text{-}i, i) = \Delta(n\text{-}j, j)] \Rightarrow \quad 2i\text{-}n = n\text{-}2j$, or
$i = n\text{-}j$.

(b) $\underline{n\text{-}2j \geq n/2;\ 2i\text{-}n \geq n/2}$

$[\Delta(n\text{-}i, i) = \Delta(n\text{-}j, j)] \Rightarrow \quad n\text{-}2i \bmod (n\text{-}1) = 2j\text{-}n \bmod (n\text{-}1)$

$2(i+j\text{-}n) \equiv 0 \mod (n\text{-}1)$

Since n-1 is odd,

$i+j\text{-}n \equiv 0 \mod (n\text{-}1)$

$i+j \equiv 0 \mod n$

Since $1 \leq i,j \leq n\text{-}1$,

$i+j = n$, or
$i = n\text{-}j$.

(c) $\underline{n\text{-}2j < n/2;\ 2i\text{-}n \geq n/2}$

$[\Delta(n\text{-}i, i) = \Delta(n\text{-}j, j)] \Rightarrow \quad n\text{-}2j = n\text{-}2i \mod (n\text{-}1)$

$2(i\text{-}j) \equiv 0 \mod (n\text{-}1)$

Since n-1 is odd,

$i\text{-}j \equiv 0 \mod (n\text{-}1)$

Since $1 \leq i,j \leq n\text{-}1$,

$i = j$.

(d) $\underline{n\text{-}2j \geq n/2;\ 2i\text{-}n < n/2}$

$[\Delta(n\text{-}i, i) = \Delta(n\text{-}j, j)] \Rightarrow \quad 2j\text{-}n \bmod (n\text{-}1) = 2i\text{-}n$

$2(i\text{-}j) \equiv 0 \mod (n\text{-}1)$

Hence (*cf.* proof for (c) above) $i = j$.

Theorem 1.6:

For any broken heteroset $i \neq n/2$, every index cycle contains at least three dominoes.

Proof:

In order for one domino to define an index cycle, it would have to be of the self-conjugate type $i\ n\text{-}i$, which occurs only in the broken heteroset $n/2$.

Assume that two dominoes are sufficient to define an index cycle. Such a cycle can be represented as $(i\ j)\ (n\text{-}j\ n\text{-}i)$, where no two of the integers i, j, $n\text{-}i$, and $n\text{-}j$ are equal. Consider the representation of the dominoes $(i\ j)$ and $(n\text{-}j\ n\text{-}i)$ as chords in the circle diagram for n. Because the two chords belong to the same heteroset, they are parallel. But Lemma 1.1 proves that it is impossible for them to be parallel if all four end points are distinct. Hence the number of dominoes in an index cycle is at least three. We prove in Theorem 1.9 that the number of dominoes in every index cycle is odd.

Example: For n=10, broken heteroset 2 is comprised of the stringlet (58)(2) and its heteroset complement, the index cycle (31)(94)(67).

Theorem 1.7:

The elements of every broken heteroset can be arranged to form one stringlet and one or more index cycles.

Proof:

$i=n/2$

The result follows immediately from the fact that the monomino $n/2$ is a stringlet and each of the $n/2$-1 dominoes is an index cycle.

$i \neq n/2$

Begin the iteration of domino placement just as if the heteroset were unbroken. Following the convention described in the proof of Theorem 1.3, place the domino that contains the index $n/2$ at the left, with the index $n/2$ in index position 1, and then add subsequent dominoes at the right, with conjugate indices in adjacent index positions but in distinct dominoes.

The iteration of domino placements is halted, before all the dominoes can be placed, by the forced placement of the monomino i, thereby creating a stringlet. (If it were not halted, the process would lead to the formation of a string, not a stringlet.)

Let P be the set of elements, including the monomino, that are contained in the singlet, and let U be the complementary set of dominoes. It is impossible for U to consist of only one domino, because that would imply that each of the two indices in P that are conjugate, respectively, to one of the two indices of that single domino occurs exactly once in P, contrary to the requirement that every index in P, aside from $n/2$, be paired with its conjugate. Hence there are at least two dominoes in U. But the set of indices in U consists of conjugate pairs, and no two of them occur in the same domino. Hence they can be arranged to form one or more index cycles. From Theorem 1.6 we conclude that the smallest of these index cycles contains at least three dominoes.

We conclude from the above arguments that the number n_s of indices in a stringlet satisfies the inequality $1 \leq n_s < n-6$. For the broken heteroset $n/2$, $n_s=1$, since for every even n, the monomino $n/2$ is a stringlet.

The distribution of broken strings, stringlet size, and the number and sizes of index cycles are described by Theorem 1.8. We denote the greatest common divisor of a and b by (a,b), and we define a *proper divisor* of the integer m to be any divisor of m—including m itself—except for 1. We denote both stringlet and string by the collective name *chain*.

Theorem 1.8

Let

s	= the heteroset number for a heteroset of (even) order n.
$\mathcal{D}(n) = \{q_1, q_2, ..., q_z\}$	= the set of proper divisors of n-1.
$\Delta(s_1, s_2)$	= the cycle distance between heterosets s_1 and s_2.
$\Delta^* = \Delta(s, n/2)$	= the cycle distance between heterosets s and $n/2$.

Then

(a) Heteroset $s = n/2$ is broken.

(b) Heteroset $s \neq n/2$ is broken iff there exists a $q_i \varepsilon \mathcal{D}(n)$ such that

$q_i \mid \Delta^*$;

otherwise, heteroset s is unbroken.

Let

$q^* = (\Delta^*, n\text{-}1)$,
n_s = the number of indices in the chain for heteroset s.
v_c = the number of index cycles in heteroset s, and
n_c = the number of indices in each index cycle in heteroset s.

Then

$$n_s = (n-1)/q^*, \tag{1.2}$$
$$v_c = (q^*-1)/2, \tag{1.3}$$
$$n_c = 2n_s \tag{1.4}$$

For two heterosets s_1, s_2 ($s_1 \neq s_2$), let

$$s = n/2 - \Delta(s_1, s_2). \tag{1.5}$$

Then heterosets s_1 and s_2 are matching iff heteroset s is unbroken.

For the broken heteroset $s = n/2$, $\Delta(s, n/2) = 0$, and $q^* = n$-1.

From Eqs. 1.2 - 1.4, for $s=n/2$,

$$n_s = 1 \tag{1.6}$$
$$v_c = n/2 - 1 \tag{1.7}$$
$$n_c = 2 \tag{1.8}$$

Example 1: Let n=46. $\mathcal{D}(46)$ = {3, 5, 9, 15, 45}. Of the 23 heterosets 1, 2, ..., 23, the following eleven are broken:

2, 3, 5, 8, 11, 13, 14, 17, 18, 20, 23.

For these heterosets, the values of $\Delta^* = \Delta(s, n/2)$ are

21, 20, 18, 15, 12, 10, 9, 6, 5, 3, 0.

respectively.

q^*	$\Delta^* = \Delta(s, 23)$	v_c	n_c	n_s
3	3 6 12 21	1	30	15
5	5 10 20	2	18	9
9	9 18	4	10	5
15	15	7	6	3
45	0	22	2	1

Example 2: Let n=64. $\mathcal{D}(64)$ = {3, 7, 9, 21, 63}. Of the 32 heterosets 1, 2, ..., 32, the following fourteen are broken:

2, 4, 5, 8, 11, 14, 17, 18, 20, 23, 25, 26, 29, 32.

For these heterosets, $\Delta^* = \Delta(s, n/2) =$

30, 28, 27, 24, 21, 18, 15, 14, 12, 9, 7, 6, 3, 0,

respectively.

q^*	$\Delta^* = \Delta(s, 32)$	v_c	n_c	n_s
3	3 6 12 15 24 30	1	42	21
7	7 14 28	3	18	9
9	9 18 27	4	14	7
21	21	10	6	3
63	0	31	2	1

Sketch of proof that $n_s = (n\text{-}1)/q^*$ for heteroset $s<n/2$ (Eq. 1.2)

Consider the two-heteroset circle diagram CD(s, $n/2$) for heterosets s and $n/2$. We can assume without loss of generality that $s<n/2$, *i.e.*, that heteroset s is a core heteroset. The cycle distance $\Delta(s, n/2)$ between heterosets s and $n/2$ is then

$$\Delta^* = n/2 - s. \tag{1.9}$$

Let

$$\delta = 2\Delta^* \tag{1.10}$$

and

$$q^* = (\Delta^*, n\text{-}1). \tag{1.11}$$

Let $\mathcal{P}$ be a connected path along the disjoint chords

$$P(1)\,P(2),\ P(2)\,P(3),\ \ldots,\ P(k)\,P(k+1),\ \ldots,$$

where $P(1) = s$, and subsequent vertices in $\mathcal{P}$ are defined as follows:

'counter-clockwise' vertex set			'clockwise' vertex set			
$P(2)$	$= (s + \delta)$	$(\mathrm{mod}\ (n\text{-}1))$	$P(3)$	$= (s - \delta)$	$(\mathrm{mod}\ (n\text{-}1))$	
$P(4)$	$= (s + 2\delta)$	$(\mathrm{mod}\ (n\text{-}1))$	$P(5)$	$= (s - 2\delta)$	$(\mathrm{mod}\ (n\text{-}1))$	
...		...				
$P(2k)$	$= (s + k\delta)$	$(\mathrm{mod}\ (n\text{-}1))$	$P(2k+1)$	$= (s - k\delta)$	$(\mathrm{mod}\ (n\text{-}1))$	(1.12)
	...			...		
$P(2m)$	$= (s + m\delta)$	$(\mathrm{mod}\ (n\text{-}1))$	$P(2m+1)$	$= (s - m\delta)$	$(\mathrm{mod}\ (n\text{-}1))$	
$P(2(m+1))$	$= (s + (m+1)\delta)$	$(\mathrm{mod}\ (n\text{-}1))$	$P(2m+3)$	$= (s - (m+1)\delta)$	$(\mathrm{mod}\ (n\text{-}1))$	
...						
$P(2(2m))$	$= (s + (2m)\delta)$	$(\mathrm{mod}\ (n\text{-}1))$	$P(2(2m+1))$	$= (s - (2m)\delta)$	$(\mathrm{mod}\ (n\text{-}1))$	

Every chord $P(2k)\,P(2k+1) = ((s + k\delta)\ (s - k\delta))\ (\mathrm{mod}\ (n\text{-}1))$ ($k = 1, 2, \ldots$) defines an element in heteroset s, since the endpoints of each such chord are located at two marked points on the circle that are on opposite sides of the marked point s and whose cycle distances from s are both equal to 2δ.

Every chord $P(2k+1)\,P(2(k+1)) = ((s - k\delta)\ ((s + (k+1)\delta))\ (\mathrm{mod}\ (n\text{-}1))$ ($k = 0, 1, 2, \ldots$) defines an element in heteroset $n/2$, since the endpoints of each such chord are located at two marked points on the circle, on opposite sides of the marked point $n/2$, whose cycle distances from $n/2$ are both equal to $(k + 1/2)\delta$. (More simply, we can prove that every such chord defines an element in heteroset $n/2$ by proving that the indices of the endpoints of the chord are conjugate:

$$\begin{aligned}(s - k\delta) + (s + (k+1)\delta)) &= 2s + \delta \\ &= 2s + 2(n/2 - s) \\ &= n.)\end{aligned} \tag{1.13}$$

The construction of $\mathcal{P}$ halts when the vertex $P(2m+1)$ in the clockwise sequence of vertices is found to coincide with the vertex $P(2m+2)$ in the counter-clockwise sequence of vertices. The vertex $P(2m+1)$ is the terminal vertex of the $(2m+1)$-th chord in $\mathcal{P}$. The coincidence of vertices $P(2m+1)$ and $P(2m+2)$ implies that

$$s - m\delta \equiv s + (m+1)\delta \pmod{(n-1)}, \tag{1.14}$$

or

$$(2m+1)\delta \equiv 0 \pmod{(n-1)}. \tag{1.15}$$

Since n-1 is odd, Congruence 1.15 implies that

$$(2m+1)\Delta^* \equiv 0 \pmod{(n-1)}. \tag{1.16}$$

The number of vertices in $\mathcal{P}$ is equal to $2m + 1$. Let us substitute n_s for $2m + 1$ in Congruence 1.16:

$$n_s\,\Delta^* \equiv 0 \pmod{(n-1)}. \tag{1.17}$$

We now invoke an elementary theorem concerning solutions of linear congruences (*cf.* Topics from the Theory of Numbers, Emil Grosswald, p. 47):

Theorem 1.9

If $(a, m) = d$, then the congruence

$$ax \equiv b \pmod{m}$$

has no solution if $d \nmid b$ and has a unique solution mod m/d if $d \mid b$.

Theorem 1.9 implies that Congruence 1.17 has a unique solution mod $\frac{n-1}{q^*}$, since $(\Delta^*, n\text{-}1) = q^*$, and $q^* \mid 0$. If we substitute

$$n_s = \frac{n-1}{q^*} \tag{1.18}$$

in Congruence 1.17, we have

$$(n-1) \mid (\Delta^* \frac{n-1}{q^*}) \tag{1.19}$$

Since $q^* \mid \Delta^*$, we conclude from Theorem 1.9 that $n_s = (n\text{-}1)/q^*$ is a unique solution mod $(n\text{-}1)/q^*$ of Congruence 1.17.

If n-1 is prime, $q^* = 1$ for every heteroset s. The construction of the path $\mathcal{P}$ does not halt until all of the n-2 chords have been traversed. $\mathcal{P}$ is therefore a string, and

$$n_s = n\text{-}1. \tag{1.20}$$

If n-1 is composite, then for every heteroset s for which $q^* = 1$, $n_s = n$-1, and $\mathcal{P}$ is a string. For every heteroset s for which $q^*>1$, Eq. 1.18 implies that $n_s<n$-1. The construction of the path $\mathcal{P}$ halts at the n_s-th vertex and thereby produces a stringlet.

We will now prove that for every $s<n/2$, the path $\mathcal{P}$ ends on the marked point $n/2$ in CD(s, $n/2$), *i.e.*, that if we substitute n_s for ($2m$+1) in the equation

$$P(2m+1) = (s - m\delta) \ (\mathrm{mod}\ (n\text{-}1)), \tag{1.12}$$

we obtain

$$P(n_s) = n/2. \tag{1.21}$$

To prove Eq. 1.21, it is sufficient to prove that

$$2P(n_s) = n \tag{1.22}$$

$$\equiv 1 \ (\mathrm{mod}\ (n\text{-}1)). \tag{1.23}$$

Substituting

$$n_s = \frac{n\text{-}1}{q^*}, \tag{1.18}$$

$$m = \frac{n_s\text{-}1}{2}, \tag{1.24}$$

and

$$s = n/2 - \Delta^*$$

$$= (n\text{-}\delta)/2 \tag{1.25}$$

in the left hand side of Eq. 1.22, we obtain

$$(2s - 2m\delta) \ (\mathrm{mod}\ (n\text{-}1) = n - \delta - \left(\frac{n\text{-}1}{q^*} - 1\right)\delta \ (\mathrm{mod}\ (n\text{-}1))$$

$$= n - \frac{(n\text{-}1)\,\delta}{q^*} \ (\mathrm{mod}\ (n\text{-}1)). \tag{1.26}$$

But $q^* \mid \delta$. Therefore

$$2P(n_s) = 2(s - m\delta) \ (\mathrm{mod}\ (n\text{-}1)) \tag{1.22'}$$

$$\equiv 1 \ (\mathrm{mod}\ (n\text{-}1)).$$

Hence

$$P(n_s) = n/2. \tag{1.21}$$

Next, in Theorem 1.10, we prove that the clockwise and counter-clockwise vertex sets defined by Eqs. 1.12 define the n_s vertices—in opposite order—of the regular star polygon $\left\{\dfrac{n_s}{(2\Delta^*/q^*)}\right\}$, which has n_s edges and 'density' (winding number) = $\dfrac{2\Delta^*}{q^*}$.

Theorem 1.10:

Consider the ccw sequence $H = [h(1), h(2), ..., h(2m+2)]$, where $m = \frac{n_s-1}{2}$, $h(1) = P(1)$, and

$$h(k) = P(2k-2) \quad (2 \le k \le 2m+2). \tag{1.27}$$

Let

$\Delta h(i) = h(i+1) - h(i)$ $(i = 1, 2, ..., 2m+1)$, and

$\delta = n - 2s.$

Then $\Delta h(i) = \delta$ $(i = 1, 2, ..., 2m+1)$. (1.28)

Proof:

According to Eqs. 1.12, the value of each term $h(i)$ in H is as follows:

1	2	3	...	k	$k+1$	...	$2m+1$	$2m+2$
$P(1)$	$P(2)$	$P(4)$	...	$P(2k-2)$	$P(2k)$	...	$P(2(2m))$	$P(2(2m+1))$
s	$s+\delta$	$s+2\delta$	...	$s+(k-1)\delta$	$s+k\delta$	...	$s+2m\delta$	$s+(2m+1)\delta$

Then

$$\begin{aligned} \Delta h(i) &= h(i+1) - h(i) \\ &= P(2i) - P(1) \\ &= \delta. \qquad (1 \le i \le 2m+1) \end{aligned} \tag{1.29}$$

From Eq. 1.10 and Congruence 1.17, it follows that

$$P(2(2m+1)) = P(1).$$

Consecutive points of H are therefore the vertices of a regular star polygon that is traced in counter-clockwise order. The angular distance between consecutive points of H is equal to $2\pi(\delta/(n-1))$. The density d (winding number) of the star polygon is therefore

$$\begin{aligned} d &= n_s \delta/(n-1) \\ &= \frac{n-1}{q^*}(2\Delta^*)/(n-1) \\ &= \frac{2\Delta^*}{q^*}. \end{aligned} \tag{1.31}$$

It is obvious from inspection of Eqs. 1.12 that a similar argument is sufficient to prove that the vertices of the same regular star polygon $\left\{\frac{n_s}{(2\Delta^*/q^*)}\right\}$ are traced in clockwise order by the clockwise sequence of vertices defined by Eqs. 1.12. We see from Eqs. 1.12 that the coincidence of the vertices defined by these two sequences is equivalent to the statement that

$$s + k\delta \equiv s - (n\text{-}1\text{-}k)\delta \pmod{(n\text{-}1)}, \tag{1.32}$$

which in turn is equivalent to the trivial congruence

$$(n\text{-}1)\,\delta \equiv 0 \pmod{(n\text{-}1)}. \tag{1.33}$$

Next we prove that for broken heteroset s,

$$n_c = 2n_s. \tag{1.6}$$

Proof: Let s be a broken heteroset, and let CD(s, $n/2$) be the two-heteroset circle diagram for heterosets s and $n/2$.

Let $N(s)$ = the number of vertices in CD(s, $n/2$) which are contained in the shorter interval between the two vertices of every closest pair of vertices in the stringlet for heteroset s.

Because the n_s vertices of the stringlet correspond to the vertices of the regular star polygon $\left\{\frac{n_s}{(2\Delta^*/q^*)}\right\}$ (*cf.* Theorem 1.10),

$$\begin{aligned} N(s) &= \delta/2\text{-}1 \\ &= \Delta^*\text{-}1. \end{aligned} \tag{1.34}$$

Now consider the sequence H_s that is comprised of the $n_s = 2m+1$ consecutive vertices of the stringlet for heteroset s (*cf.* Eqs. 1.12 and Congruence 1.34):

$$\begin{array}{ll} s & \\ (s+\delta) & \pmod{(n\text{-}1)} \\ (s-\delta) & \pmod{(n\text{-}1)} \\ (s+2\delta) & \pmod{(n\text{-}1)} \\ (s-2\delta) & \pmod{(n\text{-}1)} \\ \ldots & \\ (s+m\delta) & \pmod{(n\text{-}1)} \\ (s-m\delta) & \pmod{(n\text{-}1)}. \end{array} \tag{1.35}$$

Let α = an integer such that $1 \le \alpha \le N(s)/2$.
Consider the following directed chord sequence $DC_s{}^*$, which is derived from H_s:

$$
\begin{array}{llll}
(s - \alpha) \pmod{n-1} & (s + \alpha) \pmod{n-1}, & & \\
 & (s + \alpha) \pmod{n-1} & ((s+\delta) - \alpha) \pmod{n-1}, & \\
((s+\delta) - \alpha) \pmod{n-1} & ((s-\delta) + \alpha) \pmod{n-1}, & & \\
 & ((s-\delta) + \alpha) \pmod{n-1} & ((s+2\delta) - \alpha) \pmod{n-1}, & \\
((s+2\delta) - \alpha) \pmod{n-1} & ((s-2\delta) + \alpha) \pmod{n-1}, & & \\
 & ((s-2\delta) + \alpha) \pmod{n-1} & ((s+3\delta) - \alpha) \pmod{n-1}, & \\
\ldots, & & & \\
((s+(m-1)\delta) - \alpha) \pmod{n-1} & ((s-(m-1)\delta) + \alpha) \pmod{n-1}, & & \\
 & ((s-(m-1)\delta) + \alpha) \pmod{n-1} & ((s+m\delta) - \alpha) \pmod{n-1}, & \\
((s+m\delta) - \alpha) \pmod{n-1} & ((s-m\delta) + \alpha) \pmod{n-1}, & & \\
\end{array}
$$

- -

$$((s - m\delta) + \alpha) \pmod{n-1} \quad ((s - m\delta) - \alpha) \pmod{n-1},$$

- -

$$
\begin{array}{llll}
((s-m\delta) - \alpha) \pmod{n-1} & ((s+m\delta) + \alpha) \pmod{n-1}, & & \\
 & ((s+m\delta) + \alpha) \pmod{n-1} & ((s-(m-1)\delta) - \alpha) \pmod{n-1} & \\
((s-(m-1)\delta) - \alpha) \pmod{n-1} & ((s+(m-1)\delta) + \alpha) \pmod{n-1}, & & \\
 & ((s+(m-1)\delta) + \alpha) \pmod{n-1} & ((s-(m-2)\delta) - \alpha) \pmod{n-1}, & \\
\ldots, & & & \\
((s-2\delta) - \alpha) \pmod{n-1} & ((s+2\delta) + \alpha) \pmod{n-1}, & & \\
 & ((s+2\delta) + \alpha) \pmod{n-1} & ((s-\delta) - \alpha) \pmod{n-1} & \\
((s-\delta) - \alpha) \pmod{n-1} & ((s+\delta) + \alpha) \pmod{n-1}, & & \\
 & ((s+\delta) + \alpha) \pmod{n-1}, & (s - \alpha) \pmod{n-1}. & (1.36)
\end{array}
$$

Every chord $((s + k\delta - \alpha)\ (s - k\delta + \alpha)) \pmod{n-1}$ $(k = 0, 1, 2, \ldots, m)$ is a domino in heteroset s, since the cycle distances between each of its distinct endpoints and the marked point s are both equal to $(k\delta - \alpha)$.

Every chord $((s - k\delta - \alpha)\ (s + k\delta + \alpha)) \pmod{n-1}$ $(k = m, m-1, \ldots, 2, 1)$ is a domino in heteroset s, since the cycle distances between each of its distinct endpoints and the marked point s are both equal to $(k\delta + \alpha)$.

Every chord $((s - k\delta) + \alpha)\ ((s + (k+1)\delta) - \alpha) \pmod{n-1}$ $(k = 0, 1, 2, \ldots, m-1)$ is a domino in heteroset $n/2$, since the indices of its endpoints are conjugate:

$$((s - k\delta) + \alpha) \pmod{n-1} + ((s + (k+1)\delta) - \alpha) \equiv 2s + \delta \pmod{n-1}$$
$$= n. \qquad (1.37)$$

Every chord $((s + k\delta) + \alpha)\ ((s - (k-1)\delta) - \alpha) \pmod{n-1}$ $(k = m-1, \ldots, 2, 1)$ is a domino in heteroset $n/2$, since the indices of its endpoints are conjugate:

$$((s + k\delta) + \alpha) \pmod{n-1} + ((s - (k-1)\delta) - \alpha) \equiv 2s + \delta \pmod{n-1}$$
$$= n. \qquad (1.38)$$

The single remaining chord in DC_s*—the one between dashed lines in Eqs. 1.36—is

$$((s - m\delta)+\alpha)\ (\mathrm{mod}\ (n\text{-}1))\ \ ((s - m\delta) - \alpha)\ (\mathrm{mod}\ (n\text{-}1)).$$

This chord is a domino in heteroset $n/2$, since the indices of its endpoints are conjugate:

$$((s - m\delta)+\alpha) + ((s - m\delta)- \alpha) \equiv (2s - 2m\delta)\ (\mathrm{mod}\ (n\text{-}1)). \qquad (1.39)$$

From Congruence 1.15,

$$2m\delta \equiv \delta\ (\mathrm{mod}\ (n\text{-}1)). \qquad (1.40)$$

Substituting Congruence 1.40 on the r.h.s. of Congruence 1.39, we obtain

$$\begin{aligned}((s - m\delta)+\alpha) + ((s - m\delta)- \alpha) &\equiv (2s +\delta)\ (\mathrm{mod}\ (n\text{-}1))\\ &= n. \end{aligned} \qquad (1.41)$$

We have proved that the chord sequence DC_s* defines a closed path ('loop') composed of an alternating sequence of chords that belong respectively to heterosets s and $n/2$. Hence DC_s* is an index cycle. Since it contains two chords for each single chord of the stringlet from which it is derived,

$$n_c = 2n_s. \qquad (1.4)$$

The total number v_c of index cycles for heteroset s is readily found as follows:

Let $S(s)$ = the set of chords in the stringlet for heteroset s, and
let $S(c_i)$ = the set of chords in the i-th index cycle ($i = 1, 2, \ldots, v_c$) for heteroset s.

Since $S(s) \cap S(c_1) \cap S(c_2) \ldots \cap S(v_c) = 0$,

$$n_s + v_c(2n_s) = n\text{-}1. \qquad (1.42)$$

From Eq. 1.2,

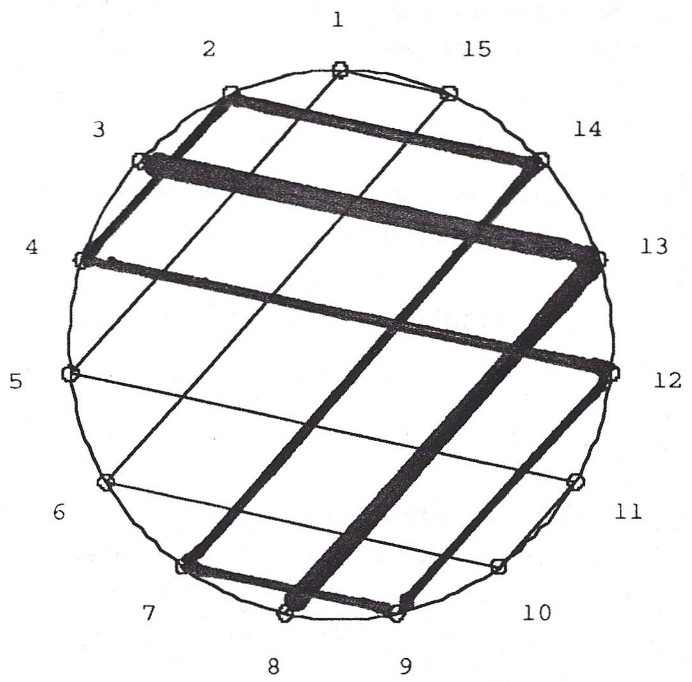
$$n_s = (n\text{-}1)/q^*. \qquad (1.2)$$

Hence

$$v_c = (q^*\text{-}1)/2. \qquad (1.3)$$

Two-heteroset circle diagrams for the broken heteroset pairs 3,8 and 5,8 for n=16 are shown on p. 19.

Two-heteroset circle diagrams for broken heteroset pairs 3,8 and 5,8, for n=16.



s=3

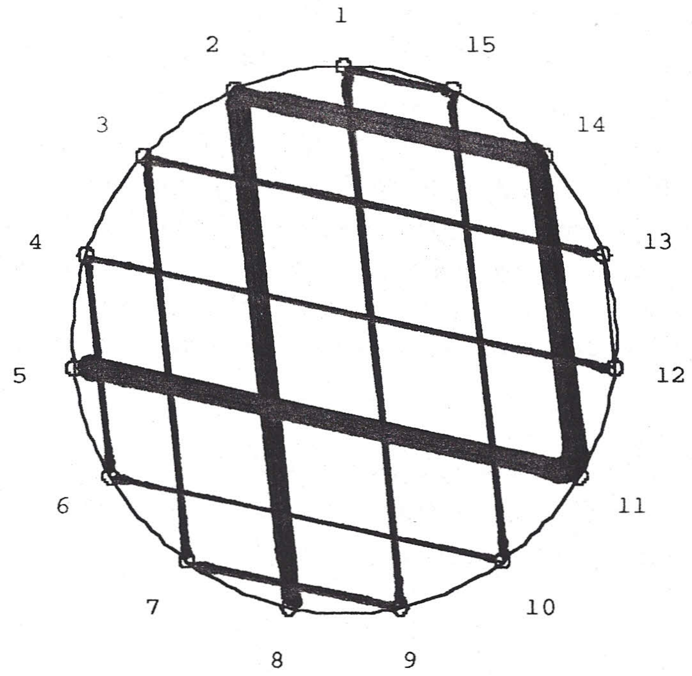
$\Delta^* = \Delta(s, n/2) = \Delta(3, 8) = 5$

$q^* = (\Delta^*, n\text{-}1) = 5$

$n_s = 3; v = 2; n_c = 6; \delta = 2\Delta^* = 10$

Stringlet: (8 13) (3)
Loops:
(*a*) (2 4) (12 9) (7 14)
(*b*) (1 5) (11 10) (6 15)

CCW regular star polygon $\left\{\frac{3}{2}\right\}$:
3, 13, 8



s=5

$\Delta^* = \Delta(s, n/2) = \Delta(5, 8) = 3$

$q^* = (\Delta^*, n\text{-}1) = 3$

$n_s = 5; v = 1; n_c = 10; \delta = 2\Delta^* = 6$

Stringlet: (8 2) (14 11) (5)
Loop:
(3 7) (9 1) (15 10) (6 4) (12 13)

CCW regular star polygon $\left\{\frac{5}{2}\right\}$:
5, 11, 2, 8, 14

We prove here that if two heterosets are matching, their respective monominoes are located at opposite ends of their common (matching) index sequence.

Theorem A.1

Let S_i denote an index sequence formed from the $n/2$ elements of heteroset i and S_j a matching index sequence formed from the $n/2$ elements of heteroset j.

Then the monominoes i and j lie at opposite ends of the index sequences S_i and S_j, respectively.

Proof

Suppose, contrary to the assertion of the theorem, that the monomino i occupies an index position in the interior of S_i. Let us denote by *end segments* the two segments of S_i—$E_1(i)$ and $E_2(i)$—that are on opposite sides of i. Both $E_1(i)$ and $E_2(i)$ are composed of dominoes only, since there is only one monomino in every heteroset. Hence the number of index positions in both $E_1(i)$ and $E_2(i)$ is even.

Now consider how the index position in S_j that corresponds to the index position occupied by the monomino i in S_i is filled. It is not filled by the monomino j, since $j \neq i$. Therefore it must be filled by the index i of a domino of heteroset j. Let us call that domino τ, and let k denote the other index of the domino τ. Let $E_1(j)$ denote the end segment of S_j whose terminal indices are k and an index at one end of S_j. The monomino j must be contained in $E_1(j)$, since the total number of index positions in $E_1(j)$ is even. Hence $E_2(j)$, like both $E_1(i)$ and $E_2(i)$, contains only dominoes. Let $E_2(i)$ be the end segment of S_i which corresponds to $E_2(j)$. It is impossible for $E_2(i)$ and $E_2(j)$ to contain the same indices in corresponding index positions, since every domino in heteroset i is different from every domino in heteroset j. We conclude that the monomino i occupies an index position not in the interior of S_i, but at one end of the index sequence S_i.

A similar argument applied to the index sequence S_j implies that the monomino j lies at one end of S_j. Since $j \neq i$, it must occupy the index position at that end of S_j which is opposite to the end of S_i occupied by the monomino i.

Corollary A.1

It is impossible for three heterosets to match.

The proof follows directly from the fact that for the monominoes i, j, and k of any three heterosets of the same order n, $i \neq j \neq k$.