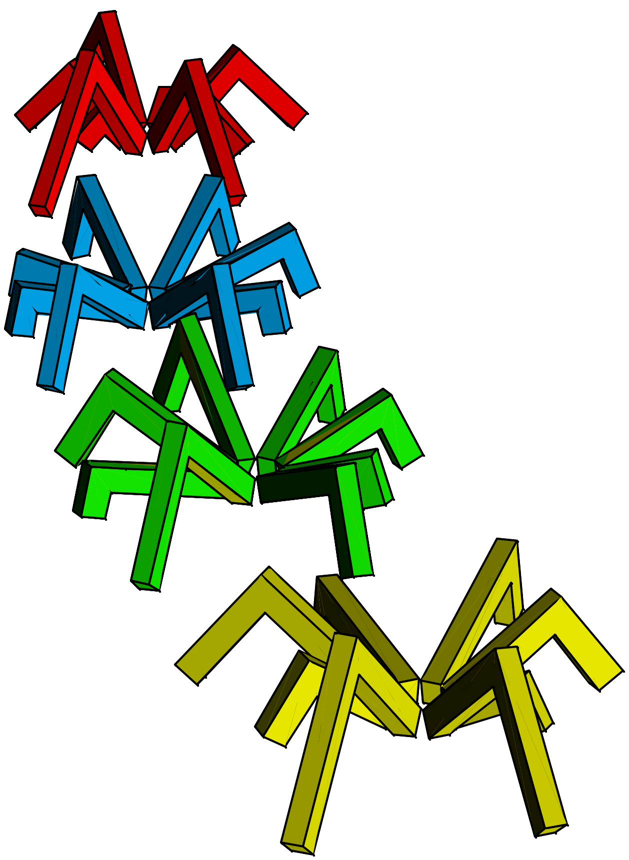
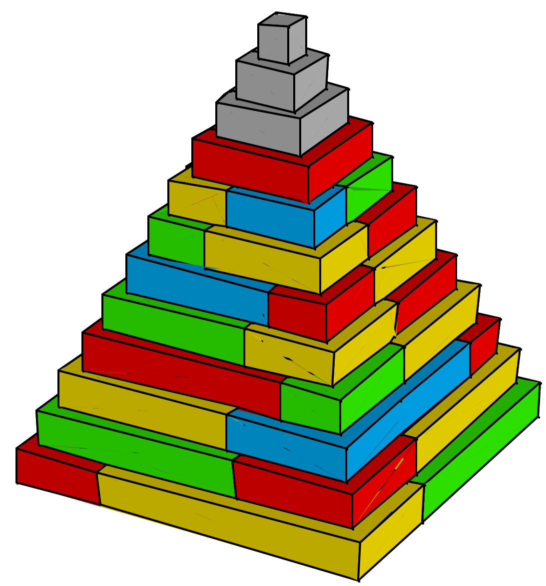
LOMINOES©

Alan H. Schoen





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PREFACE

LOMINOES are L-shaped polyominoes composed of unit cubes. Tiling and packing properties of both the standard 28-piece set L8 and a 32-piece augmented set L8† are emphasized here, but both larger and smaller sets are also described. A standard set contains one specimen of every possible size up to some designated maximum. Since the number and variety of pieces in a set are defined by simple formulas, the question of whether a set is a candidate for a particular arrangement of its pieces can sometimes be determined by mathematical analysis. The sets L8 and L8† receive special attention because the number of puzzle challenges offered by smaller sets is extremely limited, while the number and variety of pieces in larger sets make it difficult to distinguish pieces at a glance. A further reason for focusing on L8 and L8† is that eight is the smallest integer *n* for which

(a) the pieces of the standard set L*n* can be arranged in
a single ZIGGURAT or TOWER packing (*cf.* p. 30), or
a MATCHED FENCE tiling (*cf.* p. 18), or
a FILIGREE tiling (*cf.* p. 13), and
(b) the pieces of the augmented set L*n*† can be arranged in
a tiling of two congruent squares (*cf.* p. 87).

It happens also that twenty-eight**,** which is the number of LOMINOES in L8 (and the number of pieces in a standard set of Dominoes), is a *perfect* number [BEI 1966]!

But I hesitate to discourage ambitious puzzlers from exploring the challenges of larger sets of either the right-angled LOMINOES or their differently angled cousins, $_p$LOMINOES – especially packings of p-gonal $_p$TOWERS (easy) and $_p$ZIGGURATS (very difficult for $p \geq 5$), which are described on pp. 86-91 and 98-100.

The terms polyomino and pentomino were coined by the distinguished mathematician Solomon Golomb in 1954. Soon afterward Martin Gardner's columns in Scientific American introduced them to a world-wide audience. The tiling and packing properties of various sorts of polyominoes (especially pentominoes) and polycubes continue to attract new generations of enthusiasts. There is a wealth of information about both polyominoes and polycubes in *Polyominoes: Puzzles, Patterns, Problems, and Packings*, by Golomb [GOL 1994]; *Mathematical Puzzles & Diversions,* by Martin Gardner [GAR 1964]; *Polyominoes: A Guide to Puzzles and Problems in Tiling*, by George E. Martin [MAR 1996]; *Tilings and Patterns,* by Branko Grünbaum and G. C. Shephard [GRUSHE 1987]; and *O'Beirne's Hexiamond* in *The Mathemagician and Pied Puzzler*, by Richard K. Guy [GUY 1999]). The bibliography in Golomb's book lists hundreds of other articles and books about polyominoes and related topics.

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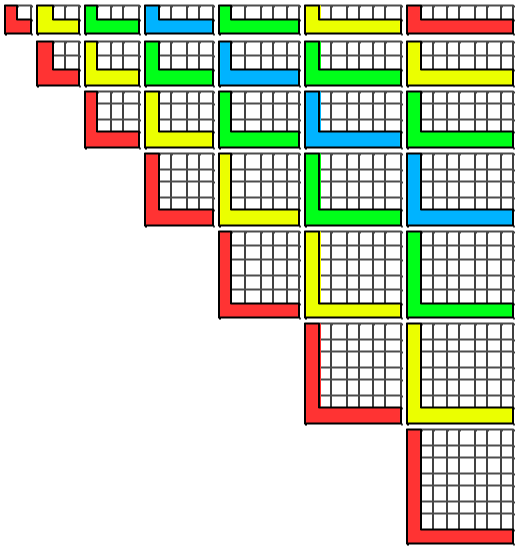
LOMINOES©

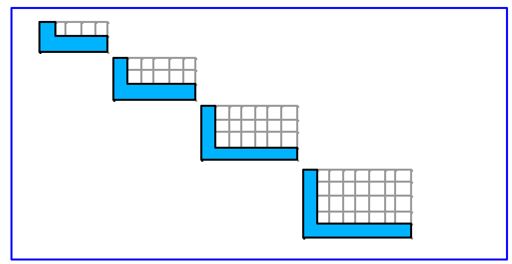


The standard set L8
arranged in the *Triangular Array*

1. The pieces of the set

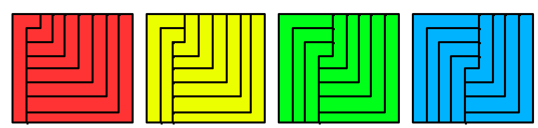
The twenty-eight LOMINOES of the *standard set* L8 (left) include one specimen of every L-shaped piece with arms of unit square cross-section that can be cut from a 1 x 8 x 8 grid of cubes.

The *augmented set* L8† contains four additional pieces (below) that are duplicates of the LOMINOES in the central NW/SE diagonal strip of L8.

L8 +  = L8†

By removing one or more columns of pieces from the right side of the *Triangular Array* (above left), one can construct any standard set of LOMINOES for $2 \leq n < 8$. Every standard set Ln of odd order n can be arranged to tile $(n-1)/2$ rectangles with proportions 1 x n x $(n+1)$; these assemblies are called *pronic rectangular subsets*. Every standard set Ln of even order n can, like L8, be transformed into the corresponding augmented set by adding duplicates of the $n/2$ pieces in the central NW/SE diagonal strip of Ln, and this augmented set can then be partitioned into $n/2$ pronic rectangular subsets.

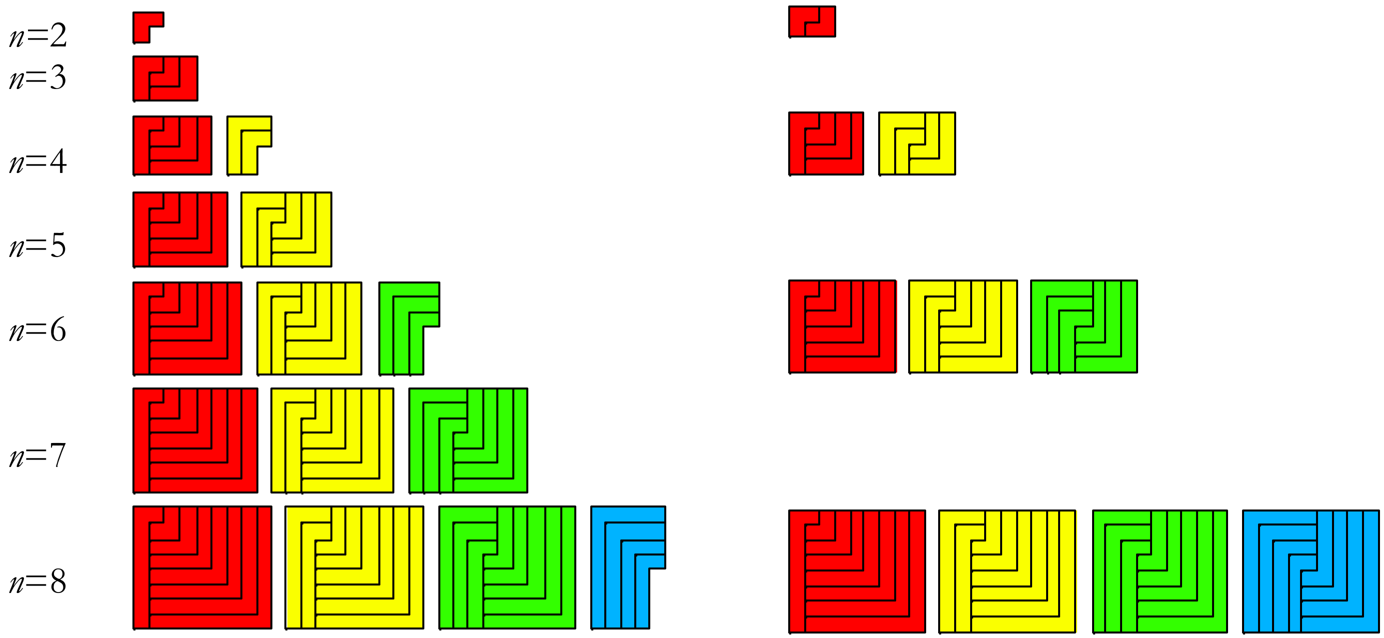
At right are the four 1 x 8 x 9 pronic subsets into which L8† can be partitioned.



1 2 3 4

LOMINOES offer a variety of puzzle challenges that are described below, but you will undoubtedly discover your own new ways to use them.

2. Canonically colored standard sets and augmented sets for $n<8$



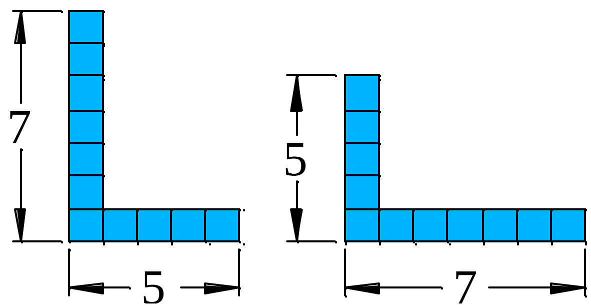
Standard sets Ln ($2 \leq n \leq 8$)   Augmented sets L$n^{\dagger}$ (n=2, 4, 6, 8)

The LOMINOES set L$8^{\dagger}$ is *canonically colored*: its four pronic subsets are colored red, yellow, green, and blue, respectively. If pieces for a smaller standard set ($n<8$) are selected from the L$8^{\dagger}$ set, that smaller set will *not* be canonically colored – the color distribution will be jumbled. However, LOMINOES is also available in a four-set version called 'LOMINOES SUPERSET' that allows the assembly of a canonically colored standard set for $2 \leq n \leq 8$. The SUPERSET contains one red, one yellow, one green, and one blue specimen of each of the thirty-two pieces in L$8^{\dagger}$.

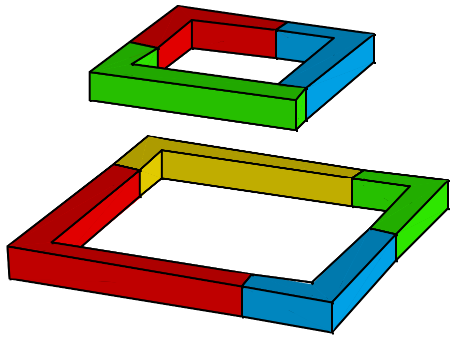
(To assemble a canonically colored *augmented* set for n=2, 4, or 6, two LOMINOES SUPERSETS are required.)

3. How the various pieces in a LOMINOES set are named

Every LOMINO is named according to the ordered pair of integers (i,j) that specifies the lengths of its two arms. Since each LOMINO may be turned over, we give it the name $[i,j]$ in one orientation and $[j,i]$ in the other. If it is placed so that it resembles the letter L, with a *stem* of length i and a *base* of length j, we name it $[i,j]$. Below (left) we show the piece corresponding to (5,7) in its two differently oriented forms. One is named [7,5] and the other [5,7].



[7, 5] **[5, 7]**
A LOMINO in two different orientations



A 3-ring of ringwidth 7
and a 4-ring of ringwidth 11

We denote a square annulus tiled by z LOMINOES as a z-*ring* ($2 \le z \le 4$). Shown above (right) are two z-rings of different overall *ringwidths* – one of ringwidth 7 and the other of ringwidth 11. Every z-ring is defined by its CCW *signature* $<[i_1, j_1][i_2, j_2][i_3, j_3][i_4, j_4]>$. The signature of the 4-ring above is <[8,3][8,8][3,6][5,3]>. For 2-rings and 3-rings, fictitious [0,1] pieces are inserted at appropriate positions in the signature to ensure that all four of the sums j_k+i_{k+1} (k=1 to 4; $i_5=i_1$) are equal to the ringwidth w. Hence the signature of the 3-ring shown above is <[3,7][0,1][6,3][4,4]>.

For the pieces of Ln ($n \ge 4$), $3 \le w \le n$ for z=2, $4 \le w \le n$ for z=3, and $6 \le w \le 2n\text{-}2$ for z=4. (3.1)
For the pieces of L$n^{\dagger}$ ($n \ge 4$), $3 \le w \le n$ for z=2, $4 \le w \le n$ for z=3, and $5 \le w \le 2n\text{-}2$ for z=4. (3.2)

There are exactly 62 ways to select four LOMINOES of L8 (or L8†) to tile a *pied* 4-ring, *i.e.,* a 4-ring composed of LOMINOES of four different colors.

4. Degenerate 4-rings

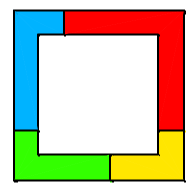
Two 4-rings composed of the same pieces are defined to be equivalent, even if the pieces in the two 4-rings are not arranged in the same order. Hence packing solutions for the ZIGGURATS and TOWERS that we will encounter on pp. 30-32 are regarded as equivalent so long as the composition of corresponding 4-rings is the same.

The signature $<[i_1, j_1][i_2, j_2][i_3, j_3][i_4, j_4]>$ of a 4-ring is a *cyclic* expression. It could equally well be written, for example, in *shifted* order, as $<[i_2, j_2][i_3, j_3][i_4, j_4][i_1, j_1]>$, or – for that matter – in *reverse* order, as $<[j_4,i_4][j_3,i_3][j_2,i_2][j_1,i_1]>$, since the pieces of a 4-ring remain in the same consecutive order when the 4-ring is either rotated or turned over. Two 4-rings composed of the same pieces are considered to define only one *arrangement* so long as the pieces in both 4-rings are in the same consecutive order.

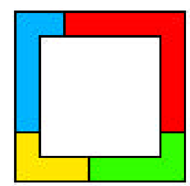
It is possible for the pieces listed in some 4-ring signatures to be rearranged to form a 4-ring with a completely different signature, *i.e.,* one that corresponds to a different cyclic sequence and therefore a different arrangement. We call a 4-ring *doubly degenerate* if the number of possible arrangements is two and *triply degenerate* if this number is three. Examples are shown at the right.

Exercise

Prove that there are no *quadruply degenerate* 4-rings.

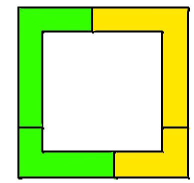


<[2,4][3,2][5,5][2,5][>

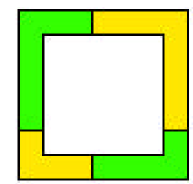


<[2,3][4,2][5,5][2,5]>

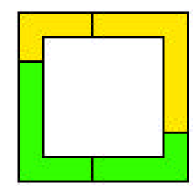
Two 4-rings with the same composition but different arrangements



<[2,4][3,2][5,4][3,5]>



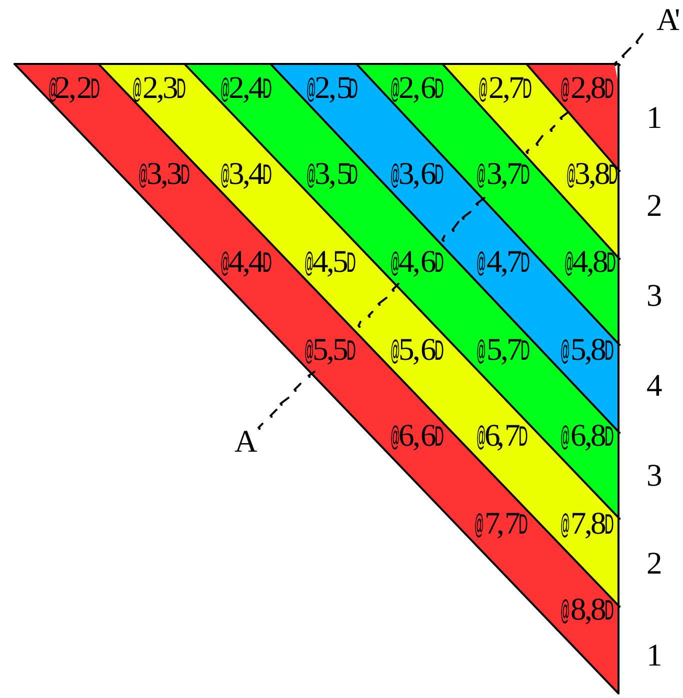
<[2,3][4,2][5,4][3,5][>



<[5,3][4,2][5,4][3,2]>

Three 4-rings with the same composition but different arrangements

5. The Triangular Array and duality



The array at the right shows the names of the pieces in the four pronic subsets of L8. The names of the subsets are the numbers listed in the column at the right of the array. For k=1, 2, and 3, subset k contains the eight LOMINOES [i,j] with j-i=k-1 or j-i=7-k. For k=4, subset 4 contains two specimens of each of the four LOMINOES [i,j] with j-i=k-1=7-k=3.

Any two LOMINOES [i,j] and [n+2-j, n+2-i] in a given subset whose name positions are related by reflection in the *medial line* AA' are called *dual*. For example, [3,4] and [6,7] are a dual pair. [2,8], [3,7], [4,6], and [5,5] are *self-dual*.

In the following pages, we discuss

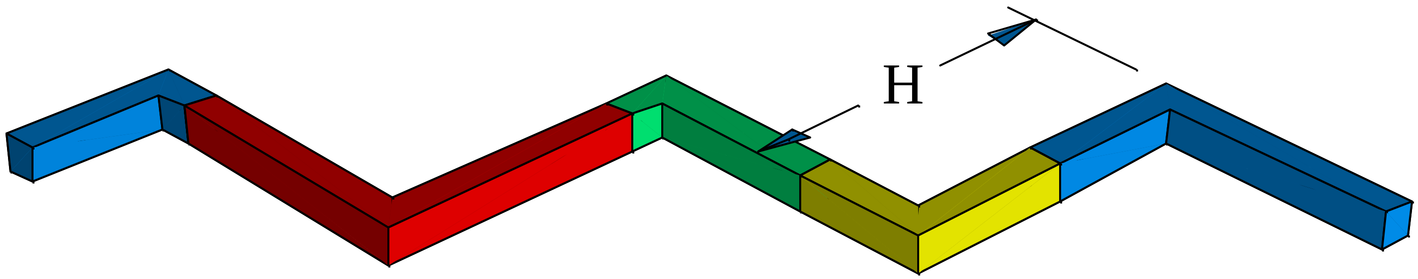
tilings of *linear*
SAWTOOTHS, **MODULATED SAWTOOTHS, FILIGREES, RUFFLES, FENCES**, and **CORRALS**;
tilings of *two-dimensional*
SQUARES, **SQUARE ANNULI**, and **RECTANGLES**; and
packings of *three-dimensional*
TOWERS, **ZIGGURATS**, and **SKYSCRAPERS.**

For every solution of a **SAWTOOTH** and also for solutions of many three-dimensional assemblies, identifying dual pairs of LOMINOES allows a wholly new *dual solution* to be derived immediately from an existing solution. Exceptions to this rule include *self-dual* packings, for which the dual solution is identical to the original solution. (Note that if the Triangular Array is expanded from the top and from the right, the dual relation can be applied even to the fictitious [0,1] LOMINOES in 2-rings and 3-rings. As a consequence, <[3,7][0,1][6,3][4,4]>, a 3-ring of ringwidth 7 (*cf.* p. 3), and <[7,3][10,9][4,7][6,6]>, a 4-ring of ringwidth 13, are duals.)

We now explore a variety of
linear patterns – SAWTOOTHS, FILIGREES, RUFFLES, FENCES, and CORRALS –
all of which are tiled by joining LOMINOES end-to-end.

6. SAWTOOTHS

A SAWTOOTH is a periodic zigzag constructed by placing all the pieces of Ln or L$n^\dagger$ (n=2,3, …) end-to-end in an alternating pattern of uniform *slant height* $H=n+2$. A five-piece segment of a complete SAWTOOTH pattern for L8 is shown below.



Although it is perhaps not immediately obvious, the pieces of Ln can always be arranged so that the slant height is uniform throughout the SAWTOOTH. Greg Martin has provided a constructive proof (*cf.* pp. 9-10). Here we simply prove that *if* a SAWTOOTH exists, the only possible value for the slant height is $n+2$:

H is the sum j_A+i_B of the lengths of contiguous arms of two adjacent pieces A=[i_A, j_A] and B=[i_B, j_B]. If $j_A=2$, the sum j_A+i_B cannot be *larger* than $n+2$, since no arm length is greater than n. On the other hand if $j_A=n$, the sum j_A+i_B cannot be *smaller* than $n+2$, since no arm length is less than 2. Hence $H=n+2$. □

Since the slant height H, which is equal to the sum of two armlengths, is equal to $n+2$, it follows that $\langle\text{armlength}\rangle_{\text{Av}}$, the average arm length of the pieces in either Ln or L$n^\dagger$, is equal to *half* the slant height, *i.e.*, $(n+2)/2$. This result can also be proved directly as follows:

Proof that the average arm length <armlength>$_{\mathrm{Av}}$ of the LOMINOES in Ln is equal to $(n+2)/2$:

The Triangular Array (*cf.* p. 5) reveals that among the LOMINOES in Ln there are exactly n arms of length k for each k in the interval $2 \leq k \leq n$. The total number of arms N_{arms} is equal to twice the number of pieces in Ln:

$$N_{arms} = 2\binom{n}{2} = n\,(n-1). \tag{6.1}$$

Hence

$$\begin{aligned}
\langle \text{armlength} \rangle_{\mathrm{Av}} &= n\,(2+3+\ \dots\ +n)/[n\,(n-1)] \\
&= n\left[\binom{n+1}{2}-1\right]/[n(n-1)] \\
&= \left[\frac{n(n+1)}{2}-1\right]/(n-1) \\
&= \frac{n^2+n-2}{2(n-1)} \\
&= (n+2)/2. \qquad \square
\end{aligned} \tag{6.2}$$

Proof that the average arm length of the pieces in L$n^{\dagger}$ is also equal to $(n+2)/2$ is left to the reader.

Two SAWTOOTH tilings are regarded as distinct if neither can be obtained from the other by a shift, reversal, or combined shift and reversal.

L3 admits only one SAWTOOTH tiling. L4 admits three, and L4† admits twenty. L5 admits two hundred and seventy-two. (No count has been made for n>5.)

CONJECTURE:
For n>3, canonically colored Ln and L$n^{\dagger}$ sets admit SAWTOOTH tilings in which no two adjoining pieces are of the same color. Such tilings are called *map-colored*.

7. Greg Martin's constructive proof that every single-set standard SAWTOOTH can be tiled

Greg Martin has proved my conjecture that a SAWTOOTH tiling exists for every standard L*n* set. We reproduce his summary of the proof here.

1. Write the LOMINOES in L*n* as ordered pairs:

$$\begin{array}{llll} [2,2], & [2,3], & \ldots & [2,n] \\ & [3,3], & \ldots & [3,n] \\ & & \ldots & \\ & & & [n,n] \end{array}$$

2. Now group these ordered pairs into the following sets:

$$\begin{array}{lllllll} [2,2], & [2,3], & \ldots, & [2,n], & [3,n], & \ldots, & [n,n]; \\ [3,3], & [3,4], & \ldots, & [3,n\text{-}1], & [4,n\text{-}1], & \ldots, & [n\text{-}1,n\text{-}1]; \\ [4,4], & [4,5], & \ldots, & [4,n\text{-}2], & [5,n\text{-}2], & \ldots, & [n\text{-}2,n\text{-}2]; \end{array}$$

and so on. The first set is all those LOMINOES with either small side 2 or large side *n*; the second set is all remaining LOMINOES with either small side 3 or large side *n*-1; the third, small side 4 or large side *n*-2; and so on. For instance, when *n*=8, the four sets are as follows: (The parentheses and commas are omitted since the numbers 2 to 8 are all single digit.)

22 23 24 25 26 27 28 38 48 58 68 78 88
33 34 35 36 37 47 57 67 77
44 45 46 56 66
55

3. Now rearrange each of these sets in the following way: start with the first LOMINO, then the last, then the second, then second-to-last, and continue shuffling in this way:

[2,2],	[n,n],	[2,3]	[n-1,n],	[2,4],	[n-2,n],	…
[3,3],	[n-1,n-1],		[3,4],	[n-2,n-1],		…

When n=8 the four sets are arranged as follows:

22 88 23 78 24 68 25 58 26 48 27 38 28
33 77 34 67 35 57 36 47 37
44 66 45 56 46
55

4. Notice that each set now forms a valid substring of the SAWTOOTH – the second number in one ordered pair plus the first number in the next ordered pair always add to n+2. All we need to do is assemble the substrings, starting from the bottom and inserting each substring into a position just before the last two LOMINOES of the substring above it. Using L8 as an example,

* first insert 55 into 44 66 45 * 56 46 in the position marked *,
  yielding 44 66 45 55 56 46;

* then insert 44 66 45 55 56 46 into 33 77 34 67 35 57 36 * 47 37,
  yielding 33 77 34 67 35 57 36 44 66 45 55 56 46 47 37;

* then insert that into 22 88 23 78 24 68 25 58 26 48 27 * 38 28 ,
  yielding 22 88 23 78 24 68 25 58 26 48 27 33 77 34 67 35 57 36 44 66 45 55 56 46 47 37 38 28.

□

8. **SAWTOOTH tiling proof for L$n^{\dagger}$**

We add to Greg Martin's construction (pp. 9-10) an insertion rule for the $n/2$ duplicate pieces

$[2, n/2+1]$, $[3, n/2+2]$, …, $[n/2+1, n]$,

read from NW to SE in the triangular array. First combine these $n/2$ pieces in pairs formed 'from the outside in': pair 1 is composed of the first and last terms, pair 2 is composed of the second and next-to-last terms, *etc.*

If n is an *evenly* even number (4, 8, 12, …), this yields the following $n/4$ pairs:

$\{[2, n/2+1], [n/2+1, n]\}$, $\{[3, n/2+2], [n/2, n-1]\}$, …, $\{[n/4+1, 3n/4], [n/4+2, 3n/4+1]\}$.

If n is an *oddly* even number (2, 6, 10, …), there are $(n-2)/4$ pairs plus one singleton:

$\{[2, n/2+1], [n/2+1, n]\}$, $\{[3, n/2+2], [n/2, n-1]\}$, …, $[(n+6)/4, (3n+2)/4]$.

In each pair the sum of the two outermost terms is equal to $n+2$. (The sum of the two terms in the singleton is also $n+2$.) Hence each pair forms a valid substring of the **SAWTOOTH**. Using L$8^{\dagger}$ for our example, let us illustrate how the **SAWTOOTH** is completed by the insertion of these pairs into the shuffled sets of Martin's step 3. We write the four duplicates as 25 36 47 58. Combining them in pairs 'from the outside in' gives

{25 58} {36 47}.

These two pairs are now inserted into the positions marked * and **.

```
22  88   *   23  78  24  68  25  58  26  48  27  38  28
33  77  **   34  67  35  57  36  47  37
44  66  45  56  46
55
```

If n is an oddly even number, a singleton is present, and it is inserted into the third position of the row following the last pair insertion row. The completed **SAWTOOTH** is then assembled exactly as in Martin's final step 4. □

9. Modulated SAWTOOTHS

{-1,0,1} two L4 sets

{-1,0,1,0} two L4 sets

{-1,-1,0,0,1,1} two L4 sets

{-1,-1,-1,-1,0,0,0,0,1,1,1,1} two L4 sets

{-1,-1,-1,0,0,0,1,1,1} three L4 sets

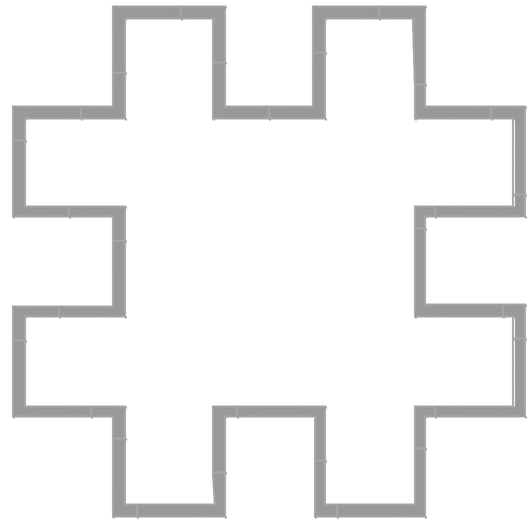
{-1,-1,0,0,1,1,0,0} four L4 sets

Six examples of modulated SAWTOOTHS for n=4

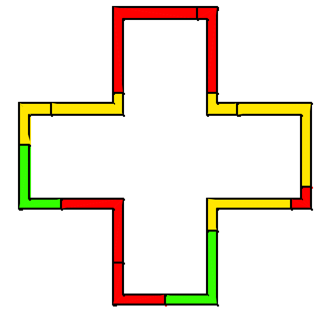
In each example, a *signal* **s** (={ }) has been added to the SAWTOOTH *carrier* **k** ={6,6,6,…}
to produce exactly one unit cell of the modulated SAWTOOTH.
The unit cell is tiled by the minimum possible number of L4 sets.

10. FILIGREES and RUFFLES

A SAWTOOTH tiled by the twenty-eight LOMINOES of L8 takes up lots of table space. It is convenient to transform it into a more compact shape by reversing the direction of the 90° turn at twelve of its twenty-eight corners. We call the resulting closed-circuit shape (right) a FILIGREE. It has square symmetry (D4 symmetry) and is the second member of an infinite sequence of crenellated *k-rings*. The i^{th} k-ring (i=1, 2, 3,…) is tiled by k LOMINOES, where $k=16i$-4. Each k-ring has i U-shaped flaps on each of its four sides. The first k-ring in this sequence (far right) is a 12-ring (*cf.* p. 43).



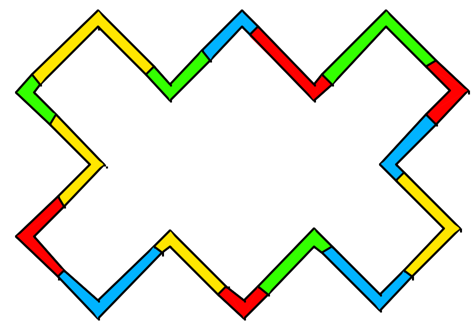
k-ring no. 2 (FILIGREE)



k-ring no. 1
(12-ring)

The number of pieces in a standard LOMINOES set Ln is $\binom{n}{2} = n(n\text{-}1)/2$.

Hence the only standard sets that contain exactly the number of LOMINOES required to tile a k-ring are those for which $n(n\text{-}1)/2=16i\text{-}4$, *i.e.,* those for which n and i are integer solutions of the Diophantine equation $n^2-n+8=32i$. The solutions are (n,i)=(8,2), (25,19), (40,49), (57,100), (72,160), (89,245}, …, *i.e.,* n=8+32j and n=25+32j (j=1, 2, …), $i=(n^2-n+8)/32$. The only FILIGREE solution that is of interest here is (n,i)=(8,2). (Can you prove that no *augmented* LOMINOES set L$n^\dagger$ tiles a k-ring?)



RUFFLE

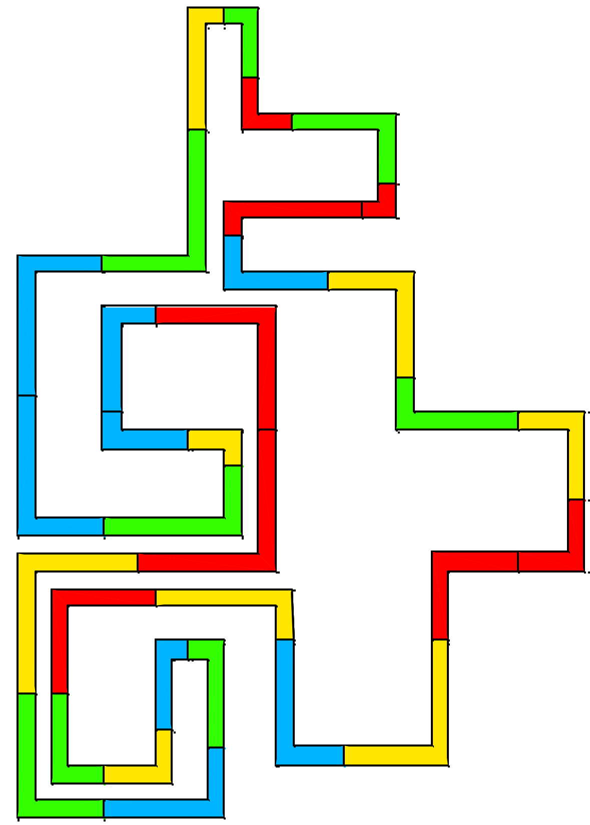
A RUFFLE, tiled by sixteen LOMINOES – four from each of the four pronic subsets of L8† – is shown above. There are only four ways to choose *two* pronic subsets of L8† that admit a RUFFLE tiling.

Of the six pairs of L8† pronic subsets – (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) – which four admit a RUFFLE tiling?

11. FENCES

FILIGREES and RUFFLES are special cases of tilings that are variously called FENCES or FARMS in the polyominoes literature [GAR 1986].

A FENCE is defined here as a flat circuit composed of all the LOMINOES of one L*n* or L*n*† set laid end-to-end with no overlaps. A FENCE is called *self-avoiding* if every piece is incident only on the two pieces at its ends and *non-self-avoiding* otherwise. The area enclosed by the self-avoiding L8† FENCE shown at the right is equal to 471.



Self-avoiding L8† FENCE

Exercises

1. Construct a self-avoiding L8 or L8† FENCE that encloses *maximum* area. (Solution is unknown.)

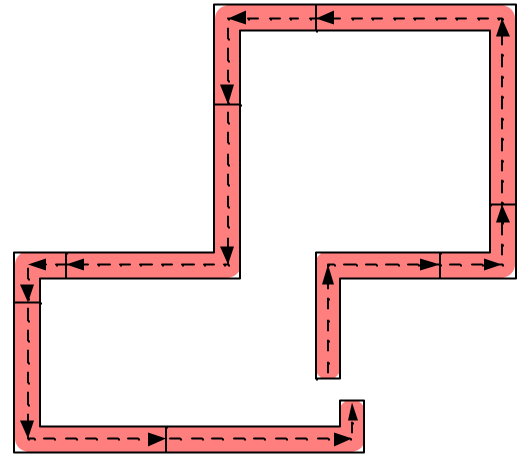
2. Construct a self-avoiding L8 or L8† FENCE that encloses *minimum* area. (Solution is unknown.)

3. Using one L8 or L8† set, construct a pair of self-avoiding FENCE-like circuits that form the inner and outer boundaries, respectively, of an annular region of *minimum* area. (Solution is known.)

4. Construct a non-self-avoiding L8 fence whose shape is [oppositely] congruent to that of its dual. (Solution is known.)

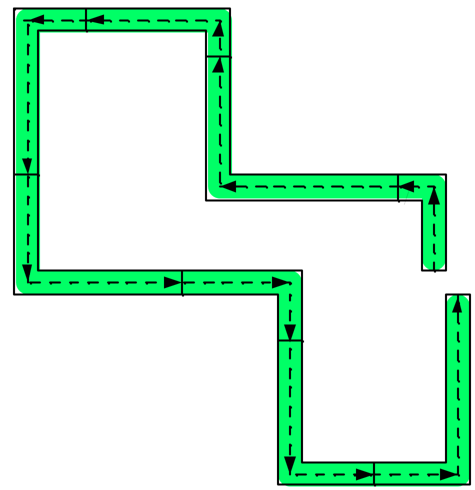
12. CORRALS

A CORRAL is a FENCE–like circuit tiled by a single pronic subset of L$n^\dagger$. At the right are two failed CORRALS tiled by L$8^\dagger$ pronic subsets 1 and 3, respectively. L$8^\dagger$ pronic subsets 1 and 3 do not admit CORRAL tilings. A proof by contradiction (below) is based on a *parity* argument. Parity proofs were first applied to the analysis of polyomino tilings by Solomon Golomb [GOL 1994].



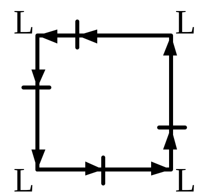
Failed FENCE tiled by L$8^\dagger$ pronic subset 1

Let us regard a CORRAL as a CCW circuit. It has a CCW *skeleton*, which is a connected sequence of sixteen vectors (*arrows*) directed along the longitudinal midlines of the arms of the LOMINOES in the CORRAL. The skeleton vectors of the two failed CORRALS are shown here as dashed arrows. The length of each dashed arrow (*reduced armlength*) is smaller by 0.5 than the length of the corresponding arm. It is convenient to analyze the closure of the skeleton instead of that of the CORRAL. If the skeleton is closed, the CORRAL is closed, and *vice versa*.



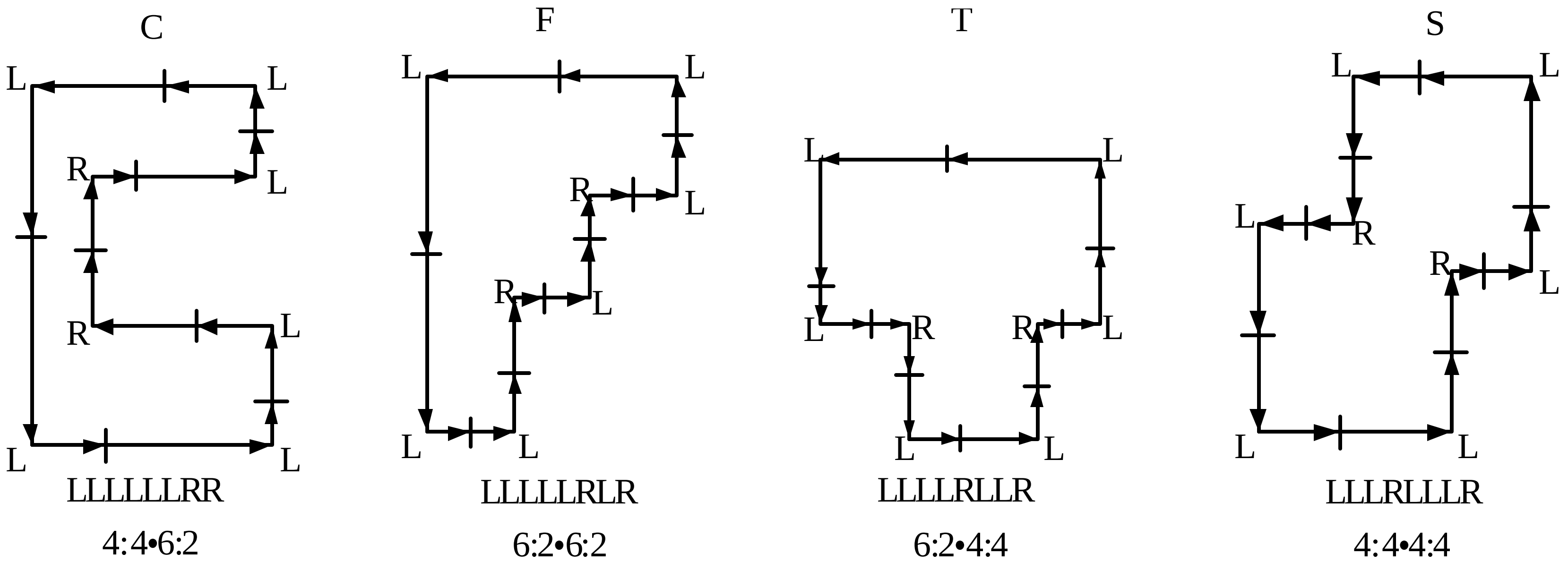
Failed FENCE tiled by L$8^\dagger$ pronic subset 3

We begin the proof by noting that a CCW circuit composed of four LOMINOES (4-ring) is based on a sequence LLLL of 90° *left turns* at the corners of the circuit (right).



A CCW circuit composed of m LOMINOES (even $m>4$) is defined by a sequence of left (L) and right (R) turns, the number of left turns exceeding the number of right turns by four. Hence for $m=8$, there are six Ls and two Rs. There are four conceivable circuit shapes for $m=8$. They are listed below and are illustrated on the next page:

-LLLLLLRR- (type C)
-LLLLLRLR- (type F)
-LLLLRLLR- (type T)
-LLLRLLLR- (type S)



If a CORRAL skeleton of type C is oriented as shown above, four of its arrows point *right*, four point *left*, six point *up*, and two point *down*. Hence we describe this configuration as 4:4/6:2. Similarly, the F-, T-, and S-type CORRALS are described as 6:2/6:2, 6:2/4:4, and 4:4/4:4, respectively. (Note that if the T-type CORRAL skeleton is rotated CCW 90°, it becomes a 4:4/6:2 configuration, like the C-type CORRAL.)

Let $S\rightarrow$ = the sum of the reduced armlengths of skeleton arrows that point *right,*
$S\leftarrow$ = the sum of the reduced armlengths of skeleton arrows that point *left*,
$S\uparrow$ = the sum of the reduced armlengths of skeleton arrows that point *up*,
$S\downarrow$ = the sum of the reduced armlengths of skeleton arrows that point *down*.

In order for a string composed of the eight LOMINOES of a pronic subset of L8[†], joined end-to-end, to define a closed circuit,

$$S\rightarrow \; = S\leftarrow \tag{12.1}$$
$$S\uparrow \; = S\downarrow \tag{12.2}$$

We classify the arms of every LOMINO [*i, j*] in a CORRAL as *horizontal* or *vertical*, according to the orientation of the LOMINO. The armlengths *i* and *j* of a single LOMINO are called *conjugate*. Conjugate *reduced armlengths* i^* and j^* are defined by

$$i^* = i - 1/2 \tag{12.3}$$
$$j^* = j - 1/2 \tag{12.4}$$

In the notation used to define the pieces of the Triangular Array (*cf.* p. 5), $i \leq j$ for every LOMINO [*i, j*]. Let us adopt this convention in listing the inventory of LOMINOES in a CORRAL. We describe the armlengths *i* and *j* in [*i, j*] as *first* and *second*, respectively, and also the reduced armlengths i^* and j^* as *first* and *second*, respectively.

Now let

$$S_{\text{horizontal}} = S_{\rightarrow} + S_{\leftarrow} \tag{12.5}$$
$$S_{\text{vertical}} = S_{\uparrow} + S_{\downarrow}. \tag{12.6}$$

Substituting from Eqs. 12.1 and 12.2 in Eqs. 12.5 and 12.6 yields

$$S_{\text{horizontal}} = 2S_{\rightarrow} \quad . \tag{12.7}$$
$$S_{\text{vertical}} = 2S_{\uparrow} \tag{12.8}$$

The LOMINOES of subset 1 are [2,2], [3,3], [4,4], [5,5], [6,6], [7,7], [8,8], [2,8]; those of subset 3 are [2,4], [3,5], [4,6], [5,7], [6,8], [2,6], [3,7], [4,8]. It is easily verified that for either subset, the sum of first reduced armlengths and the sum of second reduced armlengths are both *odd*. For a CORRAL composed of a given subset, every summand in $S_{\text{horizontal}}$ or S_{vertical} is one of two conjugate reduced armlengths for some LOMINO of the subset. Irrespective of whether a particular summand in $S_{\text{horizontal}}$ or S_{vertical} is the first or second reduced armlength of a conjugate pair, $S_{\text{horizontal}}$ and S_{vertical} are both *odd*, because for every LOMINO [*i, j*] in either subset, the reduced armlength difference $\Delta = i^* - j^*$ is *even*. This conclusion is contradicted, however, by Eqs. 12.7 and 12.8, which imply that $S_{\text{horizontal}}$ and S_{vertical} are both *even*. Hence for subsets 1 and 3 it is impossible for the CORRAL skeleton – and therefore the CORRAL itself – to be closed. □

For subsets 2 and 4, Δ is *odd*, and the sum of first reduced armlengths and the sum of second reduced armlengths are both *even*. There are at least thirty known examples of S-type CORRAL tilings by subset 2 and at least six S-, C-, and T-type tilings by subset 4.

13. MATCHED FENCES

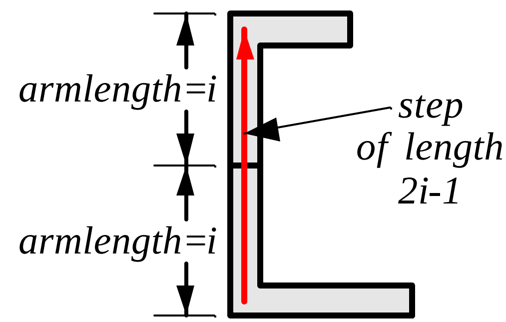
A FENCE tiled by one standard set Ln is called *matched* if the two contiguous arms of every pair of adjacent LOMINOES in the FENCE have the same armlength. In a standard set Ln there are n arms of every integer length from 2 to n inclusive. Hence in a MATCHED FENCE there are $n/2$ matched pairs of LOMINOES for each of the armlengths 2 to n. Since $n/2$ is an integer,

no MATCHED FENCE exists for odd n. (Restriction A)

Below we prove the stronger – and perhaps less obvious – restriction:

no MATCHED FENCE exists for $n \neq 8k$ (k=1, 2, …). (Restriction B)

The proof of Restriction B is based – as in Section 12 – on an analysis of the CCW skeleton of the FENCE. Consider a representative pair of matched arms, each of armlength i and therefore of reduced length $i^*=i-1/2$. We denote by l the *step* vector of length $2i^*=2i-1$ that lies on the longitudinal midline of the pair of matched arms. The FENCE skeleton is a closed chain of $\binom{n}{2}$ step vectors. Now let



A **step** (red arrow) l_k in the skeleton of a MATCHED FENCE

$$n_{horizontal} = \text{the number of horizontal steps} \quad (13.1)$$
$$n_{vertical} = \text{the number of vertical steps} \quad (13.2)$$

Then

$$n_{horizontal} = n_{\rightarrow} + n_{\leftarrow} \quad (13.3)$$
$$n_{vertical} = n_{\uparrow} + n_{\downarrow} \quad (13.4)$$

where

$$n_{\rightarrow} = \text{the number of steps that point right} \quad (13.5)$$
$$n_{\leftarrow} = \text{the number of steps that point left} \quad (13.6)$$
$$n_{\uparrow} = \text{the number of steps that point up} \quad (13.7)$$
$$n_{\downarrow} = \text{the number of steps that point down} \quad (13.8)$$

Let

$S\rightarrow$ = the sum of the lengths of the n $\rightarrow$ steps that point *right* (13.9)
$S\leftarrow$ = the sum of the lengths of the n $\leftarrow$ steps that point *left* (13.10)
$S\uparrow$ = the sum of the lengths of the n $\uparrow$ steps that point *up* (13.11)
$S\downarrow$ = the sum of the lengths of the n $\downarrow$ steps that point *down*. (13.12)

By the same arguments as in Section 12, closure of the skeleton requires that

$$S_{\text{horizontal}} = 2S\rightarrow \quad . \tag{13.13}$$

and

$$S_{\text{vertical}} = 2S\uparrow \tag{13.14}$$

The length of each of the $n_{horizontal}$ horizontal steps is *odd*, but the sum $S_{\text{horizontal}}$ of these step lengths is *even*. Hence

$$n_{horizontal} \text{ is } even.$$

Similarly, the length of each of the $n_{vertical}$ vertical steps is *odd*, but the sum S_{vertical} of these step lengths is *even*. Hence

$$n_{vertical} \text{ is } even.$$

Because the skeleton consists of an alternating sequence of horizontal and vertical steps,

$$n_{horizontal} = n_{vertical}$$
$$= \left(\frac{1}{2}\right)\binom{n}{2}. \tag{13.15}$$

Since $\left(\frac{1}{2}\right)\binom{n}{2}$ is an even integer $2m$, $\binom{n}{2}=4m$. But $\binom{n}{2}$ is divisible by four only if $n\equiv 0$ or $1 \pmod 8$. The case $n\equiv 1 \pmod 8$ is ruled out by Restriction A. Hence

$$n=8k \ (k=1, 2, \ldots) \tag{13.16}$$

□

Exercises

1. Use the eight LOMINOES of the L8† pronic subsets 2 or 4 to tile a self-avoiding CORRAL that encloses
   (a) *maximum* area;
   (b) *minimum* area.
   (It has been found that the maximum area is ≥ 228 and the minimum area is ≤ 119.)

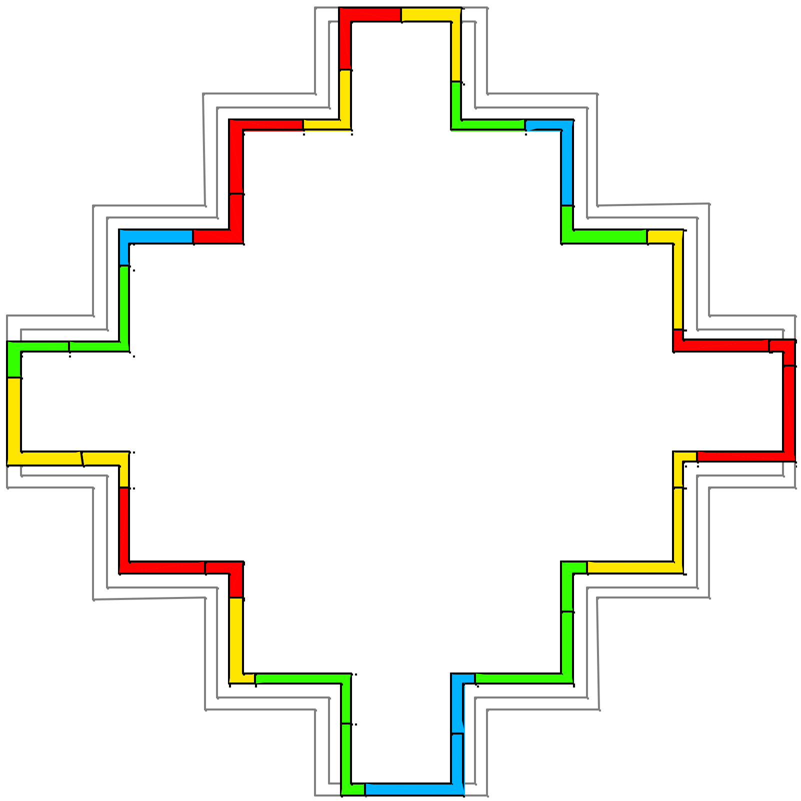
2. Prove that $\binom{n}{2}$ is divisible by four if and only if $n \equiv 0$ or 1 (mod 8).

3. What is the maximum area that can be enclosed by a L8 MATCHED FENCE? (The answer is unknown.)

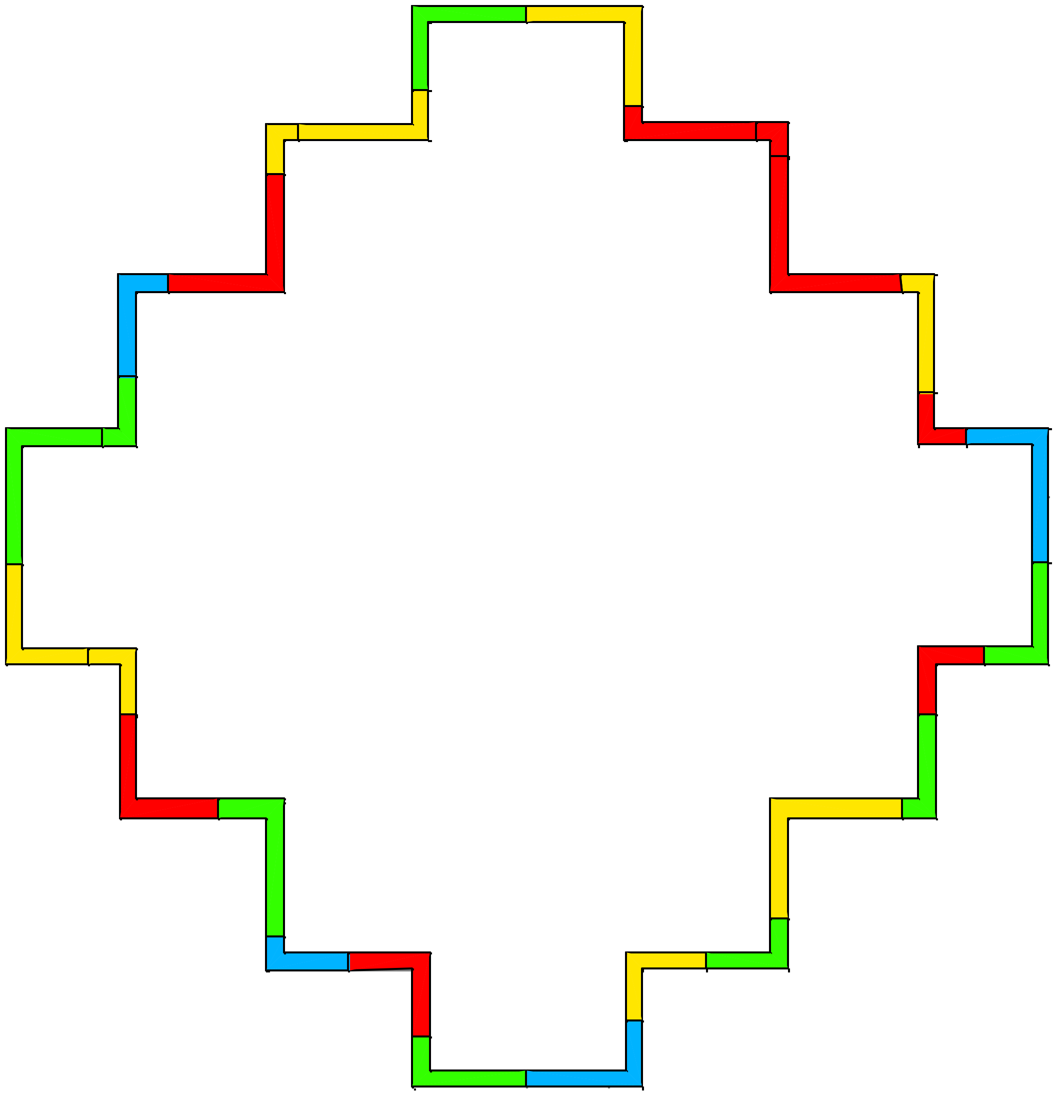
*　　*　　*　　*

Exercise 1 on p. 14 asks for a self-avoiding L8 or L8† FENCE that encloses *maximum* area. The L8 EXPANDED FILIGREE (28-ring) shown in a tiling at the right appears to be a plausible candidate for maximum area for L8. The enclosed area is 1900. This shape is obtained by folding outward the four concave U-shaped flaps at the middle N, S, E, and W positions in the FILIGREE shown on p. 13.

However, a modified version of this EXPANDED FILIGREE, which is outlined here in gray, encloses a larger area (2316) than the EXPANDED FILIGREE. A tiling of this enlarged FENCE is shown on p. 21. It is unknown whether there exists a L8 FENCE that encloses a still larger area.



L8 EXPANDED FILIGREE



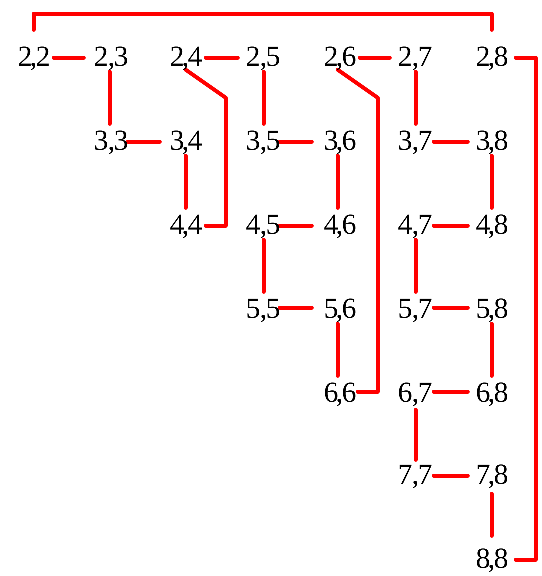
L8 modified EXPANDED FILIGREE
Enclosed area=2316

The following four-step procedure has been found to be effective in the search for MATCHED FENCE tilings by the $N(n)$ LOMINOES of Ln for n=8k (k=1, 2, …):

1. Choose a *cyclic signature* $<[a_1, a_2]\ [a_2, a_3]\ \ldots\ [a_{N\text{-}1}, a_N], [a_N, a_1]>$.
2. Apply a CCW sequence of L and R turns, with $\Sigma L = \Sigma R+4$, to the $N(n)$ consecutive LOMINOES in the signature of step 1, thereby defining the orientation of the two arms of every LOMINO in the sequence.
3. If the CCW sequence of step 2 fails to produce a closed MATCHED FENCE, replace one or more subsequences of turns (*e.g.,* LR or LRRL) with *reversed* subsequences (RL or RLLR).
4. If repeated executions of step 3 fail to produce a closed MATCHED FENCE, choose a new cyclic signature and proceed to step 2.

A 'random' algorithm for implementing step 1 (above) is as follows:

(a) Select an arbitrary LOMINO as the first piece in the signature.
(b) Choose the next piece at random from the set of matching LOMINOES that have not yet been selected.
(c) Repeat step (b) until either
  i. There are no matching pieces left and the selection halts, or
  ii. All $N(n)$ pieces are selected and a cyclic signature is defined.



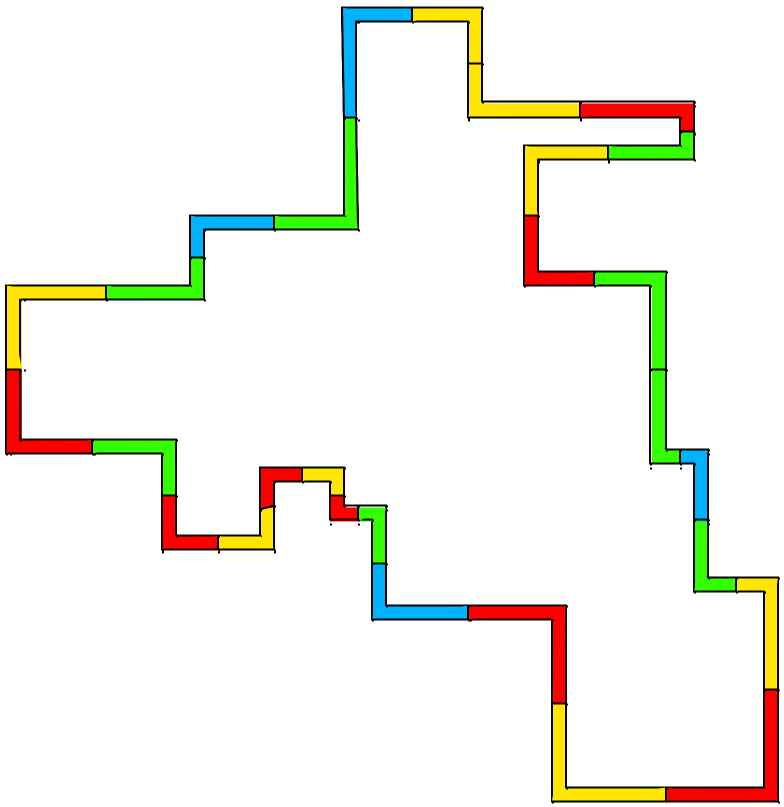
Standard cyclic signature for L8

Note: There is a trivial alternative to the random algorithm described above for implementing step 1. It is suggested by the orderly structure of the Triangular Array and is illustrated at the right. We denote the resulting cyclic signature as a *standard cyclic signature*.

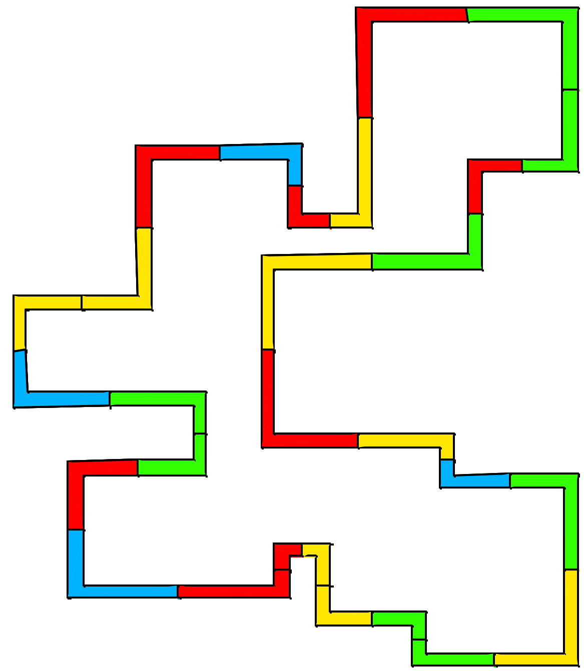
Although no MATCHED FENCE can be constructed for L6 (*cf.* Eq. 13.16), the number of cyclic signatures for L6 is very large. When the random algorithm was applied thirty-eight times to the fifteen pieces of L6, a set of thirty-eight different examples was obtained. They are listed on p. 23. The absence of repetitions in this set of thirty-eight suggests that the total number of such signatures for L6 is much larger than thirty-eight.

Random cyclic signatures for L6

solution1=[2,2],[2,3],[3,3],[3,4],[4,2],[2,5],[5,3],[3,6],[6,5],[5,5],[5,4],[4,4],[4,6],[6,6],[6,2]
solution2=[2,2],[2,3],[3,3],[3,4],[4,4],[4,6],[6,6],[6,3],[3,5],[5,2],[2,6],[6,5],[5,5],[5,4],[4,2]
solution3=[2,2],[2,3],[3,3],[3,5],[5,5],[5,6],[6,6],[6,3],[3,4],[4,4],[4,5],[5,2],[2,6],[6,4],[4,2]
solution4=[2,2],[2,3],[3,3],[3,6],[6,2],[2,5],[5,4],[4,4],[4,3],[3,5],[5,5],[5,6],[6,6],[6,4],[4,2]
solution5=[2,2],[2,3],[3,3],[3,6],[6,2],[2,5],[5,5],[5,6],[6,6],[6,4],[4,4],[4,3],[3,5],[5,4],[4,2]
solution6=[2,2],[2,3],[3,3],[3,6],[6,4],[4,4],[4,3],[3,5],[5,5],[5,2],[2,6],[6,6],[6,5],[5,4],[4,2]
solution7=[2,2],[2,3],[3,3],[3,6],[6,4],[4,4],[4,3],[3,5],[5,5],[5,2],[2,6],[6,6],[6,5],[5,4],[4,2]
solution8=[2,2],[2,3],[3,3],[3,6],[6,4],[4,4],[4,5],[5,6],[6,6],[6,2],[2,4],[4,3],[3,5],[5,5],[5,2]
solution9=[2,2],[2,3],[3,3],[3,6],[6,6],[6,4],[4,3],[3,5],[5,5],[5,4],[4,4],[4,2],[2,5],[5,6],[6,2]
solution10=[2,2],[2,3],[3,4],[4,6],[6,5],[5,5],[5,3],[3,3],[3,5],[6,6],[6,2],[2,4],[4,4],[4,5],[5,2]
solution11=[2,2],[2,3],[3,5],[5,4],[4,6],[6,6],[6,5],[5,5],[5,2],[2,4],[4,4],[4,3],[3,3],[3,6],[6,2]
solution12=[2,2],[2,3],[3,5],[5,5],[5,4],[4,4],[4,6],[6,6],[6,2],[2,5],[5,6],[6,3],[3,3],[3,4],[4,2]
solution13=[2,2],[2,3],[3,6],[6,5],[5,5],[5,3],[3,3],[3,4],[4,4],[4,2],[2,6],[6,6],[6,4],[4,5],[5,2]
solution14=[2,2],[2,3],[3,6],[6,6],[6,2],[2,5],[5,5],[5,4],[4,4],[4,3],[3,3],[3,5],[6,6],[6,4],[4,2]
solution15=[2,2],[2,3],[3,6],[6,6],[6,5],[5,3],[3,3],[3,4],[4,6],[6,2],[2,5],[5,5],[5,4],[4,4],[4,2]
solution16=[2,2],[2,4],[4,3],[3,3],[3,5],[5,6],[6,3],[3,2],[2,5],[5,5],[5,4],[4,4],[4,6],[6,6],[6,2]
solution17=[2,2],[2,4],[4,4],[4,5],[5,5],[5,6],[6,3],[3,5],[5,2],[2,6],[6,6],[6,4],[4,3],[3,3],[3,2]
solution18=[2,2],[2,4],[4,4],[4,5],[5,6],[6,6],[6,2],[2,3],[3,3],[3,4],[4,6],[6,3],[3,5],[5,5],[5,2]
solution19=[2,2],[2,4],[4,5],[5,3],[3,3],[3,4],[4,4],[4,6],[6,3],[3,2],[2,5],[5,5],[5,6],[6,6],[6,2]
solution20=[2,2],[2,4],[4,5],[5,5],[5,3],[3,3],[3,6],[6,6],[6,2],[2,3],[3,4],[4,4],[4,6],[6,5],[5,2]
solution21=[2,2],[2,4],[4,5],[5,6],[6,6],[6,2],[2,3],[3,3],[3,4],[4,4],[4,6],[6,3],[3,5],[5,5],[5,2]
solution22=[2,2],[2,4],[4,6],[6,6],[6,3],[3,3],[3,2],[2,6],[6,5],[5,3],[3,4],[4,4],[4,3],[3,5],[5,2]
solution23=[2,2],[2,4],[4,6],[6,6],[6,3],[3,3],[3,2],[2,6],[6,5],[5,3],[3,4],[4,4],[4,5],[5,5],[5,2]
solution24=[2,2],[2,5],[5,3],[3,4],[4,6],[6,5],[5,5],[5,4],[4,4],[4,2],[2,6],[6,6],[6,3],[3,3],[3,2]
solution25=[2,2],[2,5],[5,3],[3,6],[6,4],[4,3],[3,3],[3,2],[2,4],[4,4],[4,5],[5,5],[5,6],[6,6],[6,2]
solution26=[2,2],[2,5],[5,4],[4,2],[2,6],[6,6],[6,5],[5,5],[5,3],[3,3],[3,6],[6,4],[4,4],[4,3],[3,2]
solution27=[2,2],[2,5],[5,4],[4,4],[4,6],[6,6],[6,3],[3,3],[3,5],[5,5],[5,6],[6,2],[2,4],[4,3],[3,2]
solution28=[2,2],[2,5],[5,5],[5,3],[3,3],[3,6],[6,4],[4,3],[3,2],[2,6],[6,6],[6,5],[5,4],[4,4],[4,2]
solution29=[2,2],[2,5],[5,5],[5,6],[6,4],[4,5],[5,3],[3,3],[3,2],[2,6],[6,6],[6,3],[3,4],[4,4],[4,2]
solution30=[2,2],[2,5],[5,5],[5,6],[6,6],[6,3],[3,3],[3,4],[4,6],[6,2],[2,4],[4,4],[4,5],[5,3],[3,2]
solution31=[2,2],[2,5],[5,6],[6,4],[4,4],[4,5],[5,5],[5,3],[3,3],[3,6],[6,6],[6,2],[2,4],[4,3],[3,2]
solution32=[2,2],[2,6],[6,3],[3,3],[3,4],[4,4],[4,6],[6,6],[6,5],[5,4],[4,2],[2,5],[5,5],[5,3],[3,2]
solution33=[2,2],[2,6],[6,4],[4,3],[3,2],[2,4],[4,4],[4,5],[5,5],[5,6],[6,6],[6,3],[3,3],[3,5],]5,2]
solution34=[2,2],[2,6],[6,4],[4,4],[4,2],[2,3],[3,4],[4,5],[5,5],[5,6],[6,6],[6,3],[3,3],[3,5],[5,2]
solution35=[2,2],[2,6],[6,5],[5,2],[2,4],[4,3],[3,6],[6,6],[6,4],[4,4],[4,5],[5,5],[5,3],[3,3],[3,2]
solution36=[2,2],[2,6],[6,5],[5,4],[4,4],[4,6],[6,6],[6,3],[3,3],[3,4],[4,2],[2,5],[5,5],[5,3],[3,2]
solution37=[2,2],[2,6],[6,6],[6,3],[3,3],[3,2],[2,5],[5,4],[4,6],[6,5],[5,5],[5,3],[3,4],[4,4],[4,2]
solution38=[2,2],[2,6],[6,6],[6,4],[4,3],[3,2],[2,5],[5,5],[5,3],[3,3],[3,6],[6,5],[5,4],[4,4],[4,2]

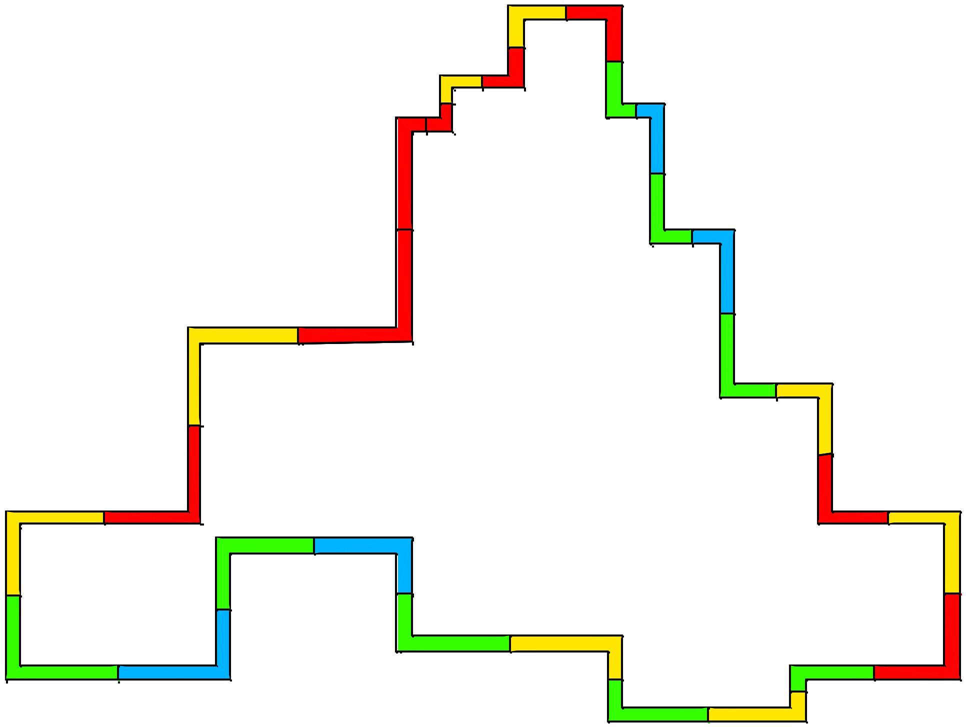


$n_{\rightarrow} = n_{\leftarrow} = n_{\uparrow} = n_{\downarrow} = 7$



$n_{\rightarrow} = n_{\leftarrow} = n_{\uparrow} = n_{\downarrow} = 7$

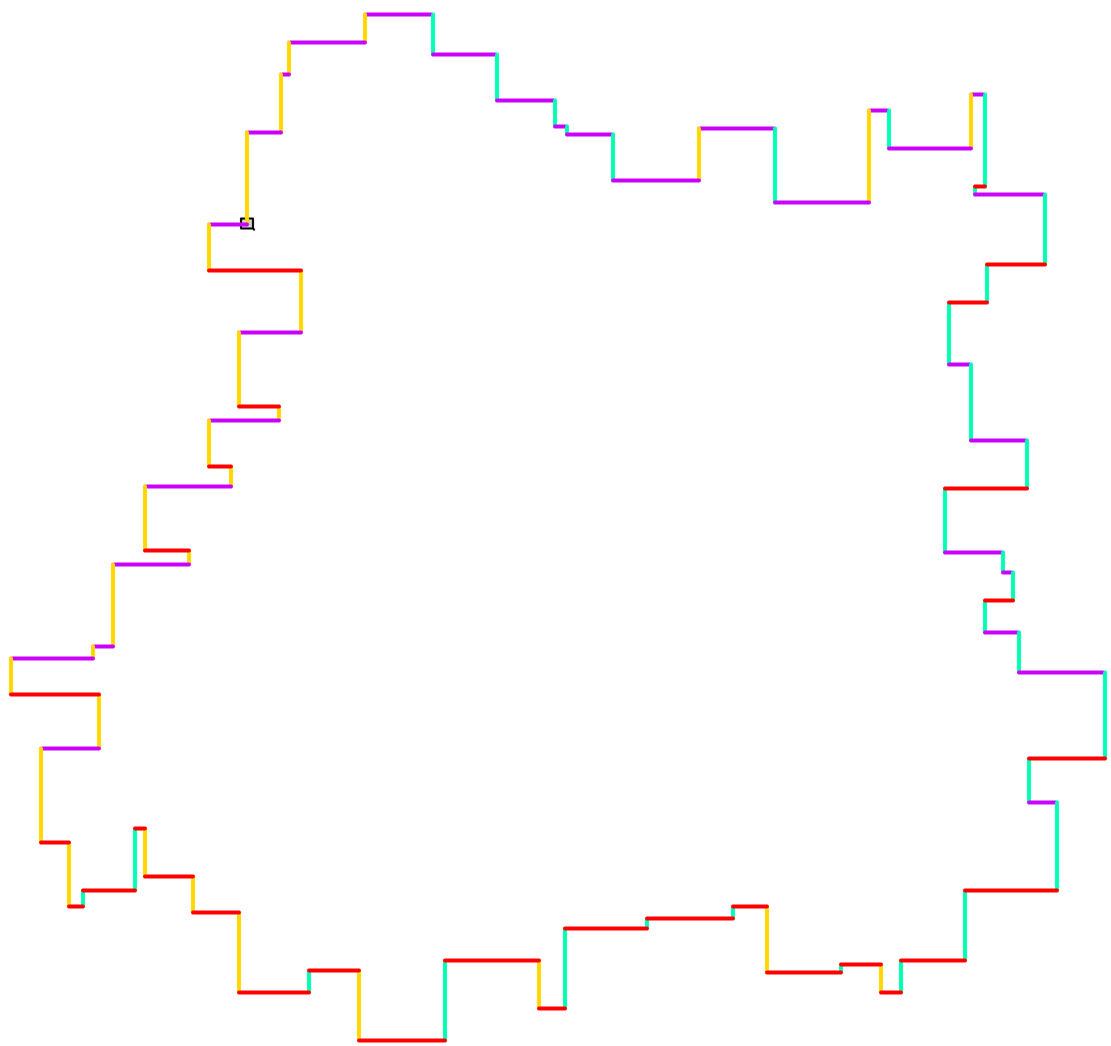
Two examples of L8 MATCHED FENCES with
random cyclic signatures



$n \rightarrow = 5$, $n \leftarrow = 9$, $n \uparrow = 7$, $n \downarrow = 7$

An example of a L8 MATCHED FENCE with the following standard cyclic signature:

[2,2] [2,8] [8,8] [8,7] [7,7] [7,6] [6,8] [8,5] [5,7] [7,4] [4,8] [8,3] [3,7] [7,2] [2,6] [6,6] [6,5] [5,5] [5,4] [4,6] [6,3] [3,5] [5,2] [2,4] [4,4] [4,3] [3,3] [3,2]



$n \rightarrow =30$, $n \leftarrow =30$, $n \uparrow =32$, $n \downarrow = 28$

Skeleton of L16 MATCHED FENCE with the random cyclic signature listed on p. 27

Random cyclic signature of the L16 MATCHED FENCE whose skeleton is shown on p. 26:

[7,8], [8, 16], [16, 11], [11, 11], [11, 13],[13, 7],[7, 3],[3, 12],[12,8],[8, 4],[4,4],[4, 15],

[15, 11],[11,8],[8,3],[3, 13],[13, 14],[14, 4],[4, 3],[3,14],[14, 6],[6, 15],[15, 10],[10, 10],[10, 16],

[16, 5],[5,11],[11,3],[3, 3],[3, 9],[9, 11],[11,2],[2,9],[9, 9],[9, 6],[6,8],[8,14],[14, 12],[12, 4],

[4, 9],[9, 12],[12, 15],[15, 14],[14,16],[16, 9],[9, 5],[5, 14],[14, 14],[14, 2],[2,15],[15,3],[3, 6],

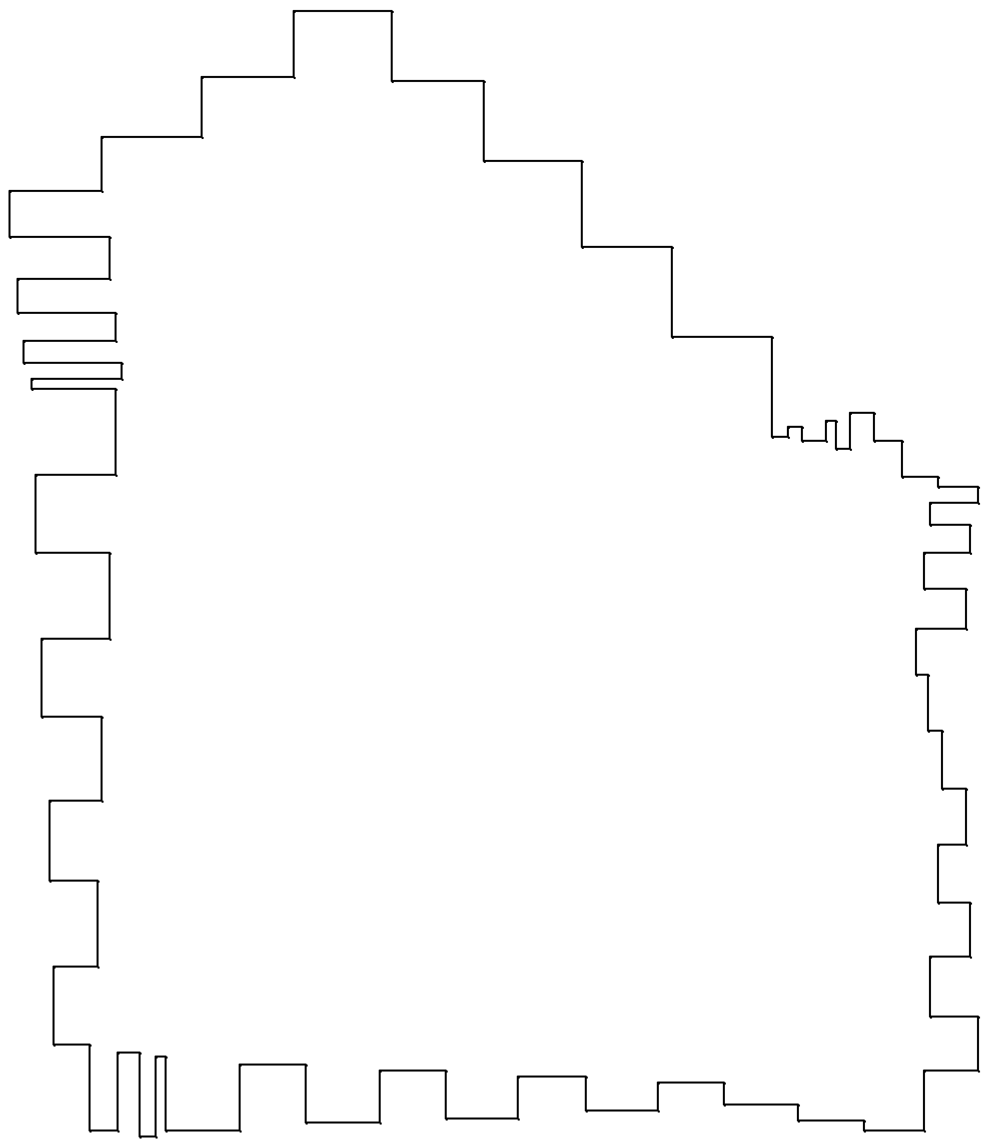
[6, 12],[12, 13],[13, 2],[2, 7],[7,5],[5, 4],[4, 6],[6, 11],[11, 12],[12,16],[16,15],[15, 5],[5,8],[8, 13],

[13, 15],[15,15],[15,7],[7, 6],[6, 6],[6, 5],[5,5],[5,2],[2, 4],[4, 10],[10, 11],[11,14],[14,9],[9, 10],

[10, 13],[13,4],[4,11],[11, 7],[7, 7],[7, 10],[10, 12],[12,12],[12,2],[2, 2],[2, 16],[16, 3],[3,10],

[10, 14],[14,7],[7,4],[4, 16],[16, 16],[16, 13],[13, 13],[13, 9],[9,15],[15,8],[8,8],[8, 2],[2,3],[3, 5],

[5, 10],[10,8],[8, 9],[9,7],[7,12],[12, 5],[5, 13],[13, 6],[6, 2],[2,10],[10, 6],[6, 16],[16, 7]



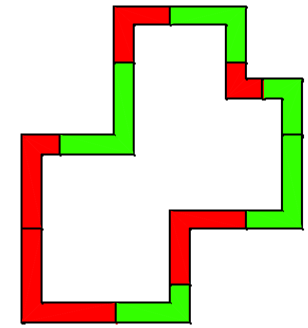
$$n \rightarrow = 29,\ n \leftarrow = 31,\ n \uparrow = 30,\ n \downarrow = 30$$

Skeleton of L16 Matched FENCE with a standard cyclic signature

Exercises

1. Tile a L4† FENCE.

2. L5 is the smallest standard set (ten LOMINOES) that tiles a FENCE.
   Find a L5 FENCE of
   (a) minimum enclosed area
   (b) maximum enclosed area
   Prove that your solutions are minimum area and maximum area, respectively.



L5 FENCE
Enclosed area = 118

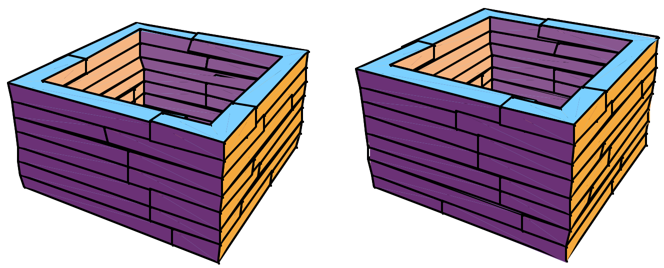
3. Let us define a MATCHED STRING, for even n, as a string composed of the $\binom{n}{2}$ LOMINOES of Ln joined end-to-end, with
   (a) cyclic signature <$[a_1, a_2]$ $[a_2, a_3]$ … $[a_{N-1}, a_N]$, $[a_N, a_1]$> and
   (b) L and R turns in alternating sequence L R L R L … R L R.

   Explain why for even $n \geq 4$, the number of SAWTOOTH tilings is greater than the number of MATCHED STRING tilings. (Hint: Carefully examine the case n=4, for which there are three SAWTOOTH tilings but only one MATCHED STRING tiling.)

4. Apply an analysis similar to that on pp. 15-17 to determine which pronic subsets of Ln†, for even n>8, cannot tile a CORRAL.

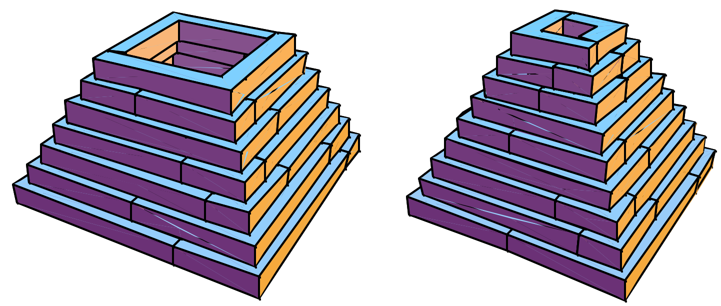
14. TOWERS and ZIGGURATS

LOMINOES' principal 3D puzzle challenges are the packing of two related kinds of structures – **TOWERS** and **ZIGGURATS**. Below are two examples of each of these shapes, identified by names that are explained on pp. 32-33. Each ZIGGURAT and one of the TOWERS is composed of one L8 set; the taller TOWER requires the augmented set L8†. (The precise arrangement of pieces shown for each structure is fictitious, *i.e.,* different from that in any actual packing.)



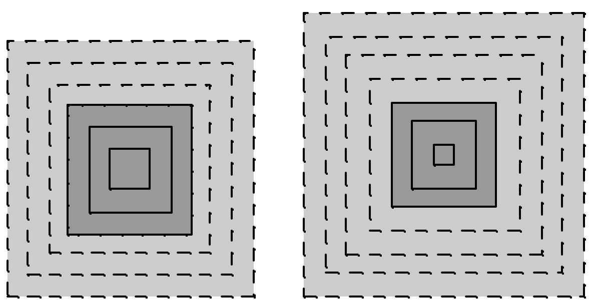
$T_1[8\,|\,10,10]_7$ $T_1[8^\dagger\,|\,10,10]_8$

Two examples of TOWERS

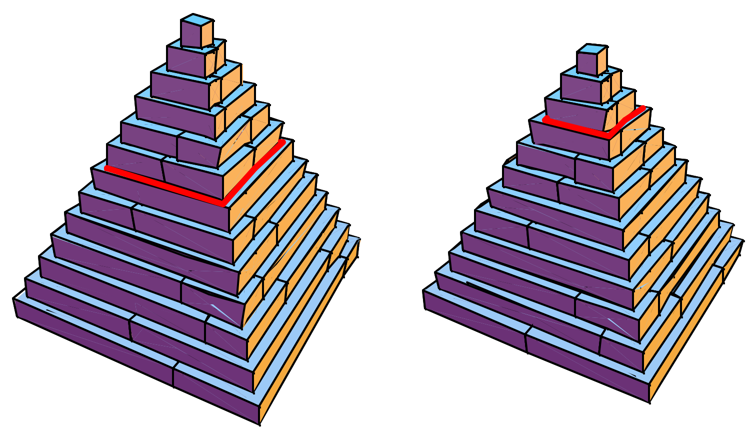


${}_1[8\,|\,7,13]_1$ ${}_1[8\,|\,4,12]_1$

Two examples of ZIGGURATS



Dark gray CAP RINGS (*even* at left, *odd* at right) surrounded by the 4-rings (**light** gray) of ${}_1[8\,|\,7,13]_1$



Capped ZIGGURATS (*stepped pyramids*)

Each of the two ZIGGURATS shown on p. 30 is converted into a stepped pyramid if the gray *cap rings* included with the LOMINOES set are placed on top. These square annular rings serve also as *tiling templates* in the search for ZIGGURAT packings. For example, *even* cap rings of ringwidths 2, 4, and 6 serve as tiling cores for 4-rings of ringwidths 4, 6, and 8, respectively, while *odd* cap rings of ringwidths 1, 3, and 5 are used for tiling 4-rings of ringwidths 3, 5, and 7, respectively (*cf.* illustrations at lower left on p. 30). A ZIGGURAT is constructed by first assembling the *z*-rings of odd and even ringwidths in separate nested tilings and then stacking these *z*-rings in an alternating sequence.

15. ZIGGURAT Vital statistics

For a ZIGGURAT to have a packing by Ln it is necessary that $V_{\text{set}}(n)=V_{\text{rings}}(a,b)$.
For a ZIGGURAT to have a packing by L$n^\dagger$, it is necessary that $V^\dagger_{\text{set}}(n)=V_{\text{rings}}(a,b)$.

$$\boxed{N(n)= n(n\text{-}1)/2 = \text{number of LOMINOES in L}n} \qquad (15.1)$$

$$\boxed{N^\dagger(n)= 2(n/2)^2 = \text{number of LOMINOES in L}n^\dagger} \qquad (15.2)$$

$$\boxed{V_{\text{set}}(n)=n(n^2\text{-}1)/2=\text{volume of L}n.} \qquad (15.3)$$

$$\boxed{V^\dagger_{\text{set}}(n)=n^2(n+1)/2=\text{volume of L}n^\dagger} \qquad (15.4)$$

$$\boxed{\begin{array}{c}\text{Volume } V_{\text{rings}}(a,b) \text{ of } \boldsymbol{r}\ z\text{-rings of ringwidths } a,\ a+1,\ldots,\ b\text{-}1,\ b \\ =2(\boldsymbol{b}-\boldsymbol{a}+1)(\boldsymbol{a}+\boldsymbol{b}), \\ \text{where } \boldsymbol{r}=\boldsymbol{b}-\boldsymbol{a}+1,\ \boldsymbol{a}=a\text{-}1, \text{ and } \boldsymbol{b}=b\text{-}1.\end{array}} \qquad (15.5)$$

Exercise

Derive Eqs. 15.1-15.4. (Eq. 15.5 is derived on p. 45.)

16. How ZIGGURATS are named

A ZIGGURAT is a *hollow stepped truncated pyramid* composed of a collimated stack of r z-rings whose ringwidths w assume consecutive integer values from a to b (inclusive). Hence $r=b-a+1$. A ZIGGURAT is called *regular* if every one of its z-rings is a 4-ring; otherwise it is called *irregular*. A single ZIGGURAT $_q[n|a,b]_1$ composed of the pieces of q Ln sets ($q \geq 1$) is called *singular*. A singular ZIGGURAT for which q=1 is called *solitary*. Both $_1[8|7,13]_1$ (regular) and $_1[8|4,12]_1$ (irregular) are examples of solitary ZIGGURATS.

A ZIGGURAT **COMPLEX** $_q[n|a,b]_s$ is a set of s ZIGGURATS, composed of the pieces of q Ln sets ($q \geq 1$, $s \geq 1$), whose external shapes are congruent. Examples of ZIGGURAT COMPLEXES are shown on pp. 54, 57-60. A solitary ZIGGURAT is a particular instance of a ZIGGURAT COMPLEX for which q=s=1.

The ZIGGURAT COMPLEX $_q[n|a,b]_s$ is described as of type $q{:}s$. If it is *regular*, then

$$q = lcm[V_{\text{set}}(n), V_{\text{rings}}(a,b)] / V_{\text{set}}(n) \qquad (16.1)$$
$$s = lcm[V_{\text{set}}(n), V_{\text{rings}}(a,b)] / V_{\text{rings}}(n) \qquad (16.2)$$

For each of the ZIGGURATS of a *regular* ZIGGURAT COMPLEX, r is odd (*cf.* pp. 65-66). Two infinite families of *irregular* ZIGGURAT COMPLEX candidates are known; for one (*cf.* p. 54), r is odd; for the other (*cf.* pp. 82-85), r is even. It is unknown whether there exists a third infinite family to which the solitary irregular ZIGGURAT $_1[11|4,19]_1$ belongs (*cf.* p. 34).

Exercise

Derive Eqs. 16.1 and 16.2.

17. How TOWERS are named

A standard TOWER $T_q[n|n+2,n+2]_s$ is a collimated stack composed of the s 4-rings of the regular ZIGGURAT COMPLEX $_q[n|n+2,n+2]_s$; q is the smallest number of Ln sets for which $N(n)$ is divisible by four.

$$q = lcm[N(n),4]/N(n) \qquad (17.1)$$

$$s = q\ N(n)/4 \qquad (17.2)$$

Similarly, for an augmented TOWER $T_{q\dagger}[n^\dagger|n+2,n+2]_{s\dagger}$,

$$q^\dagger = lcm[N^\dagger(n),4]/N^\dagger(n) \qquad (17.3)$$

$$s^\dagger = q^\dagger\ N^\dagger(n)/4 \qquad (17.4)$$

A TOWER is called *singular* if it is composed of the pieces of q standard sets or $q^\dagger$ augmented sets (q, $q^\dagger \geq 1$), and a singular TOWER is called *solitary* if q or $q^\dagger = 1$. Solitary standard TOWERS are defined for $n \equiv 0$ or 1 (mod 8). Solitary augmented TOWERS are defined for $n \equiv 0$ (mod 4).

Exercise

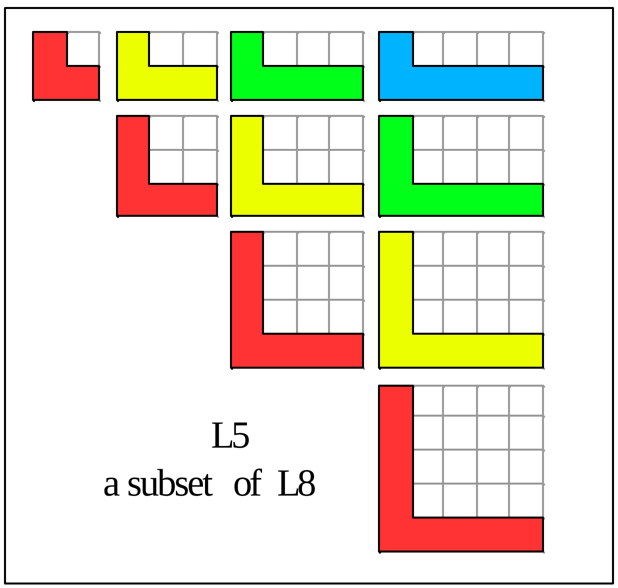
Derive Eqs. 17.1-17.4.

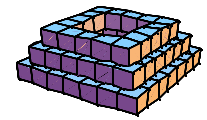
18. There are five standard ZIGGURATS of type 1:1
(*solitary ZIGGURATS composed of* **one** *Ln set*).

These five ZIGGURATS are:

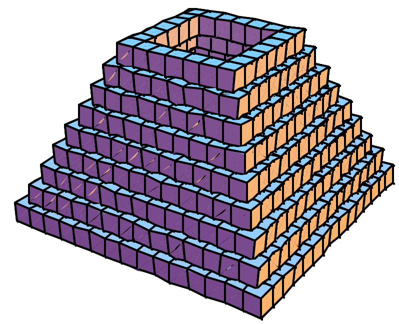
$_1[5\,|\,5,7]_1$	irregular	ZIGGURAT	(1 solution)	
$_1[8\,|\,4,12]_1$	irregular	ZIGGURAT	(59 solutions)	(*cf.* p. 30)
$_1[8\,|\,7,13]_1$	regular	ZIGGURAT	(384 solutions)	(*cf.* p. 30)
$_1[9\,|\,7,15]_1$	regular	ZIGGURAT	(6772 solutions)	
$_1[11\,|\,5,19]_1$	irregular	ZIGGURAT	*	

$_1[5\,|\,5,7]_1$ is an excellent choice for a ZIGGURAT warm-up exercise. Below are the ten pieces of L5:

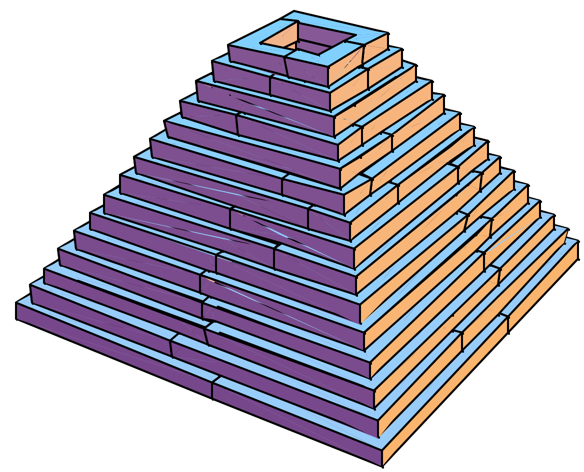




$_1[5\,|\,5,7]_1$



$_1[9\,|\,7,15]_1$



$[11\,|\,5,19]_1$
(Fictitious packing)

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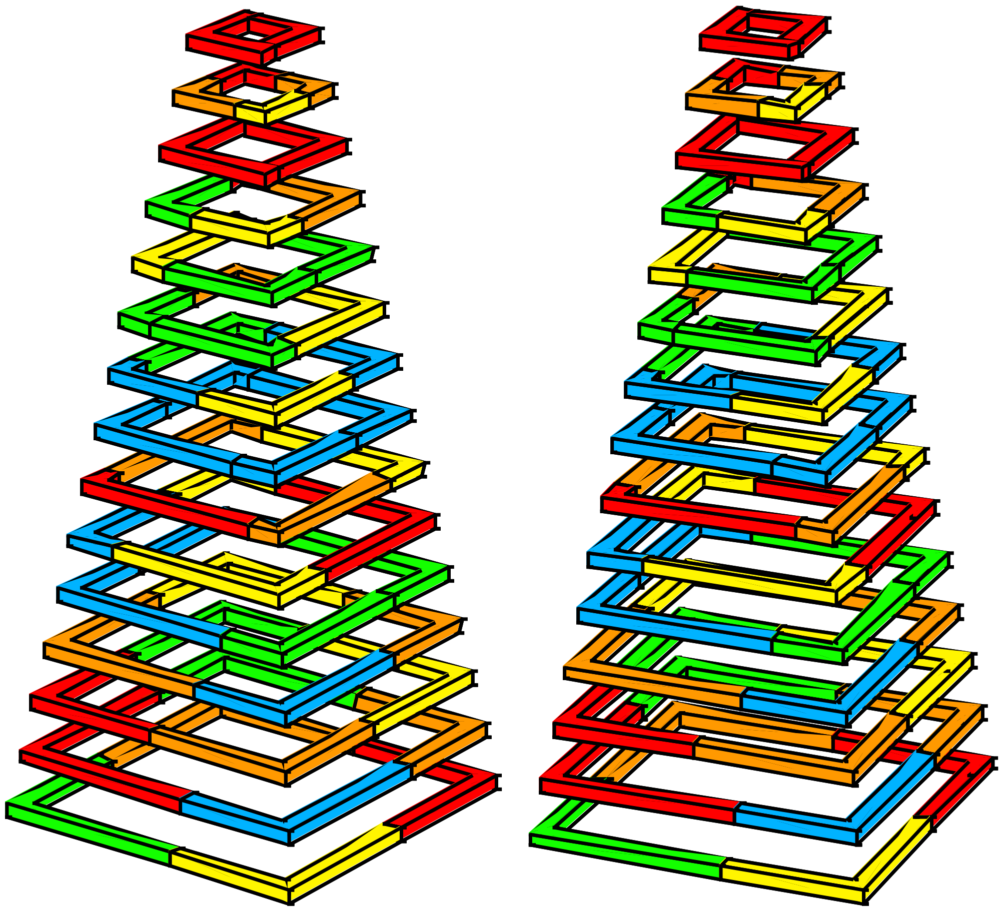
*There is one known packing for $_1[11\,|\,5,19]_1$.
The total number of packings is unknown.



[11;5,19] ziggurat

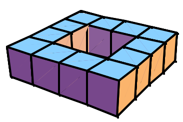
19. There are four standard ZIGGURATS of type q:1 with $q>1$ (singular ZIGGURATS composed of two or more Ln sets).

${}_4[2\,|\,4,4]_1$	regular	ZIGGURAT	(1 solution)
${}_4[3\,|\,4,6]_1$	regular	ZIGGURAT	(1 solution)
${}_2[4\,|\,5,7]_1$	regular	ZIGGURAT	(1 solution)
${}_2[5\,|\,5,9]_1$	regular	ZIGGURAT	(3 solutions)

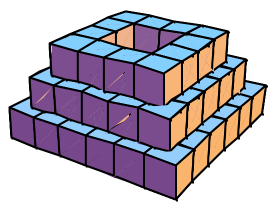
${}_4[2\,|\,4,4]_1$, shown below, can be described as a one-story TOWER/ZIGGURAT/SKYSCRAPER.

${}_2[4\,|\,5,7]_1$, which is composed of the twelve LOMINOES of *two* L4 sets, has the same external shape as ${}_1[5\,|\,5,7]_1$ (*cf.* p. 34), which is composed of the ten LOMINOES of *one* L5 set.

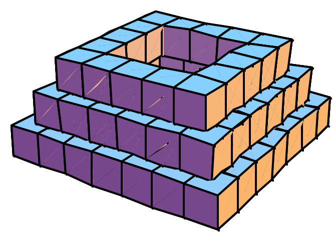
Two of the three packings of ${}_2[5\,|\,5,9]_1$ are dual; the third packing is self-dual (*cf.* p. 5).



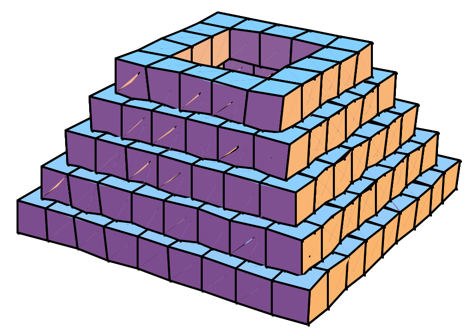
${}_4[2\,|\,4,4]_1$



${}_4[3\,|\,4,6]_1$



${}_2[4\,|\,5,7]_1$



${}_2[5\,|\,5,9]_1$

Exercise

Prove that the irregular ZIGGURAT ${}_1[5\,|\,5,7]_1$ has only one packing solution.

20. TOW, an algorithm for packing a self-dual solitary standard TOWER

We now describe an algorithm called TOW that exploits the symmetry of the Triangular Array to generate a self-dual packing of any solitary standard TOWER $T_1[n\,|\,n+2,n+2]_s$ (*cf.* pp. 33, 107). Augmented TOWERS can be packed either by a slightly modified version of TOW or else by a scheme that is sketched in Exercise 4 on p. 125. Here we will illustrate TOW for $n=8$ and $n=16$, but it is valid for all $n \equiv 0$ or 1 (mod 8).

We recall that the only LOMINOES sets that admit a packing of a one-set TOWER are those for which the number of pieces in the set, $N(n)=n(n-1)/2$ (standard set) or $N^{\dagger}(n)=2(n/2)^2$ (augmented set), is divisible by four:

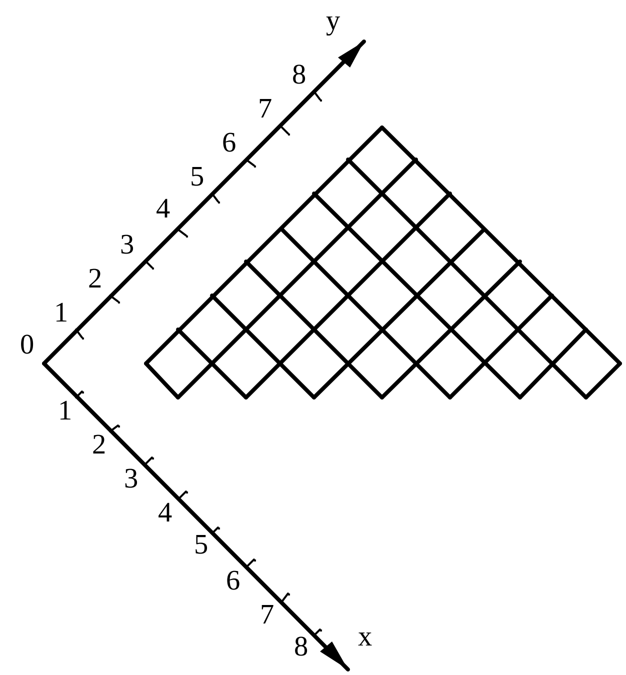
L8, L9, L16, L17, L24, L25, … (standard sets)
L4, L8, L12, … (augmented sets)

Below we describe TOW for (a) L8 and (b) L16.

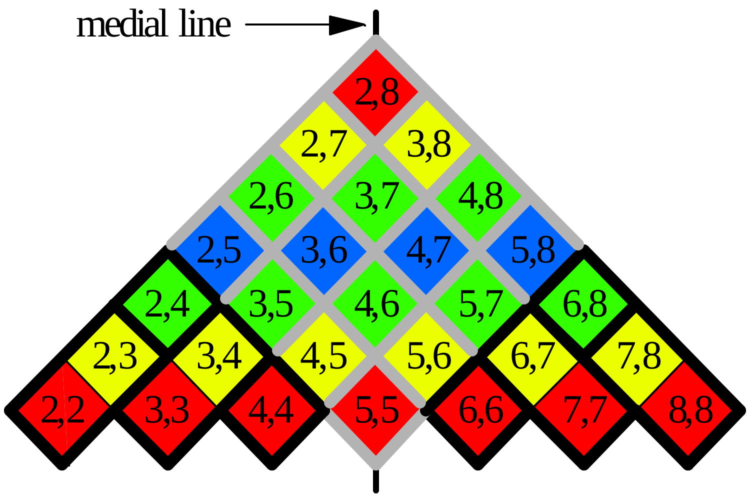
a. Packing an L8 TOWER by TOW

Let us attach a system of rectangular coordinate axes for the *continuous* variables (x,y) to the space of *discrete* row-and-column indices in the Triangular Array (*cf.* p. 5). The array is shown at the right rotated 45° counter-clockwise to emphasize its bilateral symmetry. At the point (i,j) in row i and column j,

$$x=i \qquad (20.1)$$
$$y=j \qquad (20.2)$$

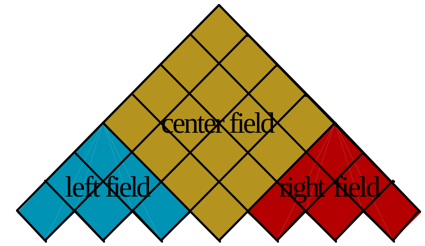


rotated triangular array



L8 triangular array
Four *dominoes* and two *monominoes*
in left and right fields are outlined in black.

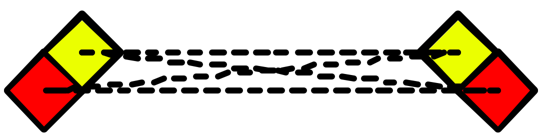
The Triangular Array for L8 is composed of three disjoint regions denoted as *center field*, *left field*, and *right field*. In center field (below right) we trace four concentric square circuits that connect single squares (here called **monominoes**) with dashed lines, thereby defining four 4-rings of ringwidth ten.



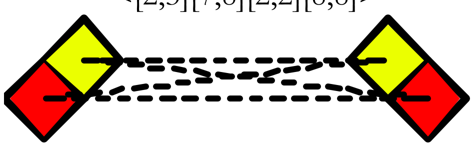
The three fields
of the L8 triangular array

The dashed lines that join squares in the four two-square **dominoes** in left and right field (below left) define two 4-rings of ringwidth ten. The two outer dominoes, {[2,2],[2,3]} and {[7,8],[8,8]}, are related by reflection in the medial line of the triangular array (*cf.* p. 5), and the two inner dominoes, {[3,3],[3,4]} and {[6,7],[7,7]}, are similarly related.

Finally, the two dual pairs of monominoes in left and right field – {[2,4],[6,8]} and{[4,4],[6,6]} – may be combined to define a seventh 4-ring of ringwidth ten. □

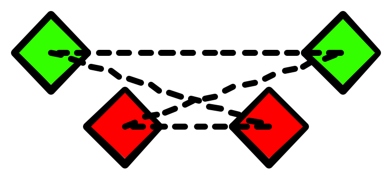


<[2,3][7,8][2,2][8,8]>



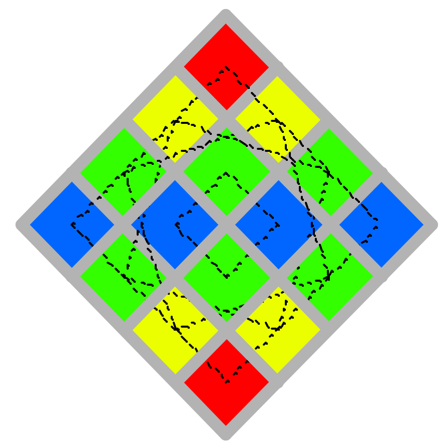
<[3,4][6,7][3,3][7,7]>

The two domino-based 4-rings
in left field/right field



<[2,4][6,6][4,4][6,8]>

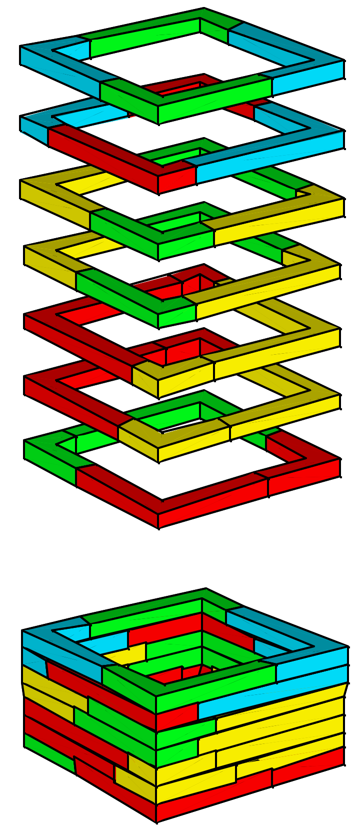
The single monomino-based 4-ring
in left field/right field



<[3,7][3,6][4,6][4,7]>
<[2,8][2,5][5,5][5,8]>
<[2,7][3,5][5,6][4,8]>
<[2,6][4,5][5,7][3,8]>

The four monomino-based 4-rings
in center field

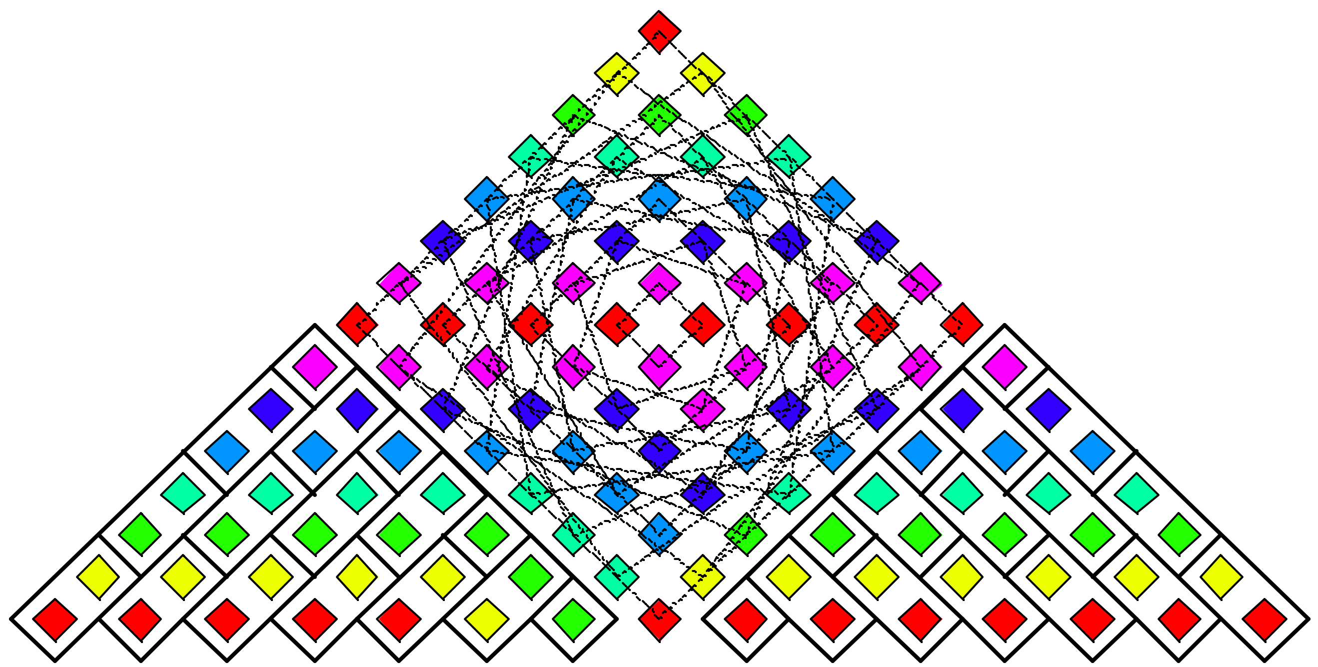
The L8 TOWER assembled from these seven 4-rings is shown at the right.



TOWER $T_1[8|10,10]_7$ derived from TOW

b. Packing an L16 TOWER by TOW

The circuits in the Triangular Array below define the composition of the 4-rings in center field. The monominoes and dominoes in left and right fields are joined in circuits analogous to those described on p. 37 for L8. (The details are left to the reader.)



L16 triangular array

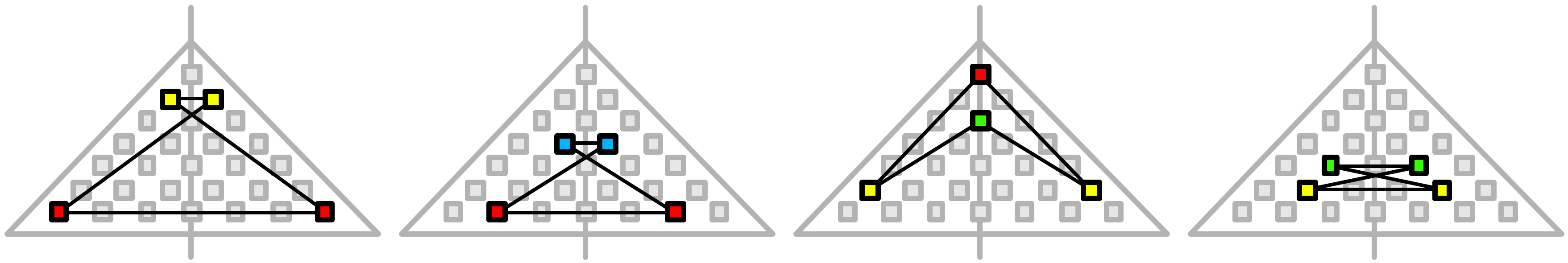
Sixteen 4-rings are formed out of the sixty-four *monominoes* in center field.
Twelve 4-rings are formed out of twenty-four pairs of *dominoes* in left and right fields.
Two 4-rings are formed out of eight *monominoes* in left and right fields.

21. A packing of the TOWER $T_1[8|10,10]_7$ that is *not* derived by TOW

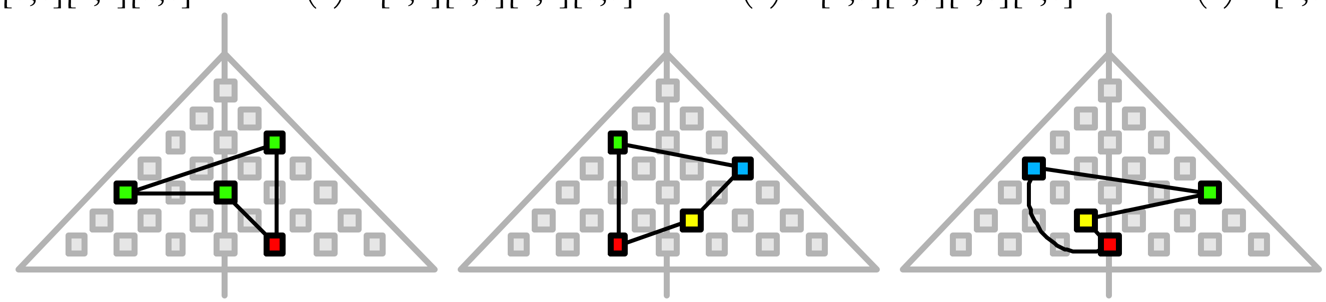
Now let us consider an example of a packing of the solitary TOWER $T_1[8|10,10]_7$ that is *not* derived by the TOW algorithm. Its 4-ring signatures are listed at the right. The circuits in the Triangular Array that define 4-rings 1, 2, 3, and 4 (below) are symmetrical by reflection in the medial line of the Triangular Array, but those defined by 4-rings 5, 6, and 7 have no symmetry. In many examples of packings of the L8 standard TOWER, *every* 4-ring circuit has bilateral (D1) symmetry and is therefore self-dual. In such a case, the TOWER packing itself is called self-dual.

The seven 4-rings of a randomly selected packing of the TOWER $T_1[8|10,10]_7$

(1) <[2,2][8,8][2,7][3,8]>
(2) <[7,7][3,3][7,4][6,3]>
(3) <[2,3][7,3][7,8][2,8]>
(4) <[4,3][7,5][5,3][7,6]>
(5) <[4,2][8,4][6,6][4,6]>
(6) <[5,8][2,6][4,4][6,5]>
(7) <[4,5][5,5][5,2][8,6]>



(1) <[2,2][8,8][2,7][3,8]> (2) <[7,7][3,3][7,4][6,3]> (3) <[2,3][7,3][7,8][2,8]> (4) <[4,3][7,5][5,3][7,6]>



(5) <[4,2][8,4][6,6][4,6]> (6) <[5,8][2,6][4,4][6,5]> (7) <[4,5][5,5][5,2][8,6]>

The seven 4-ring circuits of a 'non-TOW' packing of the TOWER $T_1[8|10,10]_7$

22. The centrum of a 4-ring

Let us define the *centrum* of a 4-ring with signature <$[i_1,j_1][i_2,j_2][i_3,j_3][i_4,j_4]$> as

$$(x_c,y_c)=\left(\frac{1}{4}\right)\sum_{k=1}^{4}(i_k\,,\,j_k)\,. \tag{22.1}$$

The centrum of a 4-ring is the centroid of the four points in the Triangular Array whose rectangular coordinates (x,y) are respectively equal to the row and column indices of the four LOMINOES of the 4-ring (*cf.* Eqs. 20.1 and 20.2).

The centrum of every 4-ring of ringwidth n+2 lies on the medial line of the Triangular Array.

Proof:

The medial line is the locus of points (x,y) in the Triangular Array for which

$$x+y=n+2 \tag{22.2}$$

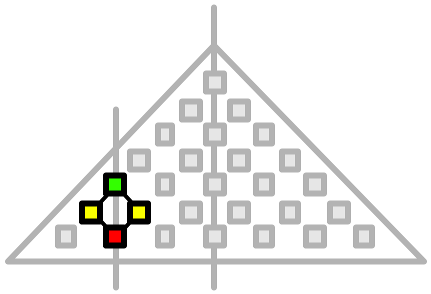
The signature of a 4-ring of ringwidth n+2 is <$[n+2-d,a][n+2-a,b][n+2-b,c][n+2-c,d]$>, where each of the positive integers a, b, c, d lies in the interval $2 \le integer \le n$. Let $u=(a+b+c+d)/4$. Then the centrum of the 4-ring is $(x,y)=(n+2-u,\ u)$ and therefore lies on the medial line. (Note that u is not necessarily an integer.) □

Let us define a *w-paramedial* line in the Triangular Array as the locus of points (x,y) for which $x+y=w$ ($6 \le w \le 2n-2$; $w \ne n+2$). We leave to the reader the proof that

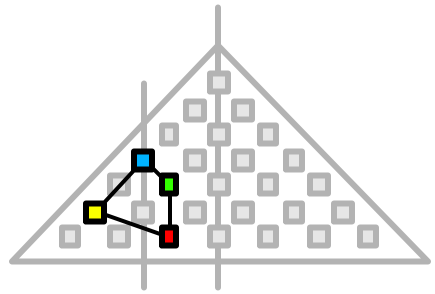
(a) every w-paramedial line is parallel to the medial line, and
(b) the centrum of a 4-ring of width w lies on the w-paramedial line.

On p. 41 we show examples of L8 4-rings of every possible ringwidth w in the interval $6 \le w \le 14$.

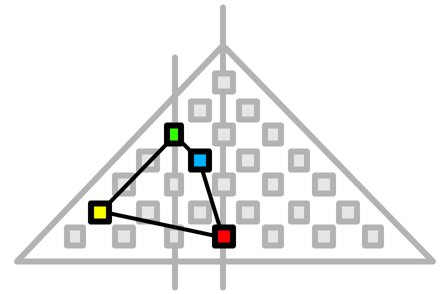
23. Examples of L8 4-rings of every possible ringwidth



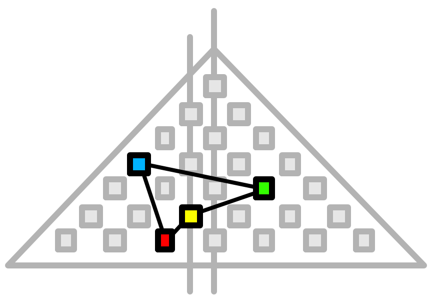
w=6: <[2,4][2,3][3,3][3,4]



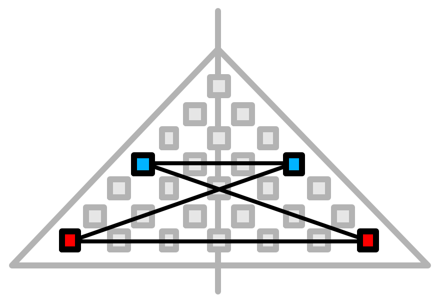
w=7: <[2,5][2,3][4,4][3,5]



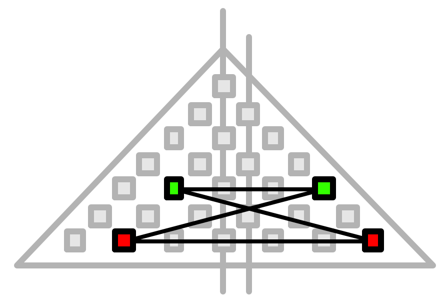
w=8: <[2,6][2,3][5,5][3,6]



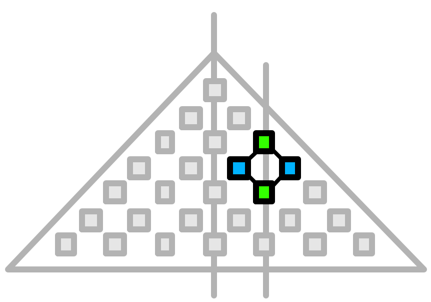
w=9: <[2,5][4,4][5,4][5,7]



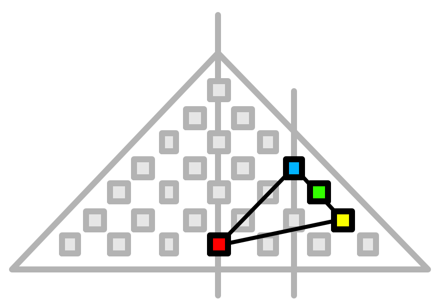
w=10: <[2,5][5,8][2,2][8,8]



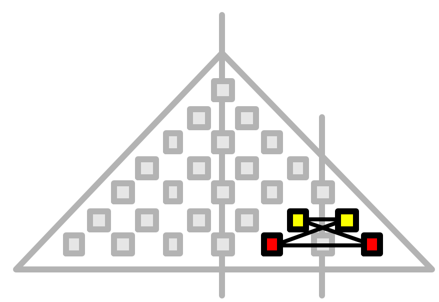
w=11: <[3,5][6,8][3,3][8,8]



w=12: <[4,8][4,7][5,7][5,8]



w=13: <[5,8][5,5][8,7][6,8]



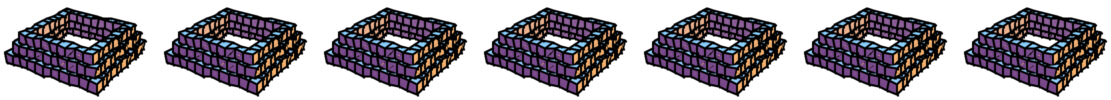
w=14: <[6,6][8,8][6,7][7,8]

Arbitrarily chosen examples of L8 4-rings of every possible ringwidth w in the interval $6 \leq w \leq 14$

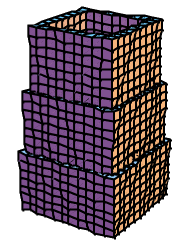
24. SKYSCRAPERS

A **SKYSCRAPER** $S_q[n\,|\,a,b]_s$ is a collimated assembly formed by stacking the $s\,r$ z-rings contained in the ZIGGURAT COMPLEX $_q[n\,|\,a,b]_s$ in r $(=b-a+1)$ tiers, each of which contains s z-rings of the same ringwidth. The ringwidth in each tier is greater by one than the ringwidth in the tier immediately above.

$S_3[8\,|\,9,11]_7$ (below right) is an example of a three-tier regular SKYSCRAPER. The first step in its construction is the arrangement of the eighty-four LOMINOES of three L8 sets to form the regular ZIGGURAT COMPLEX $_3[8\,|\,9,11]_7$ (below).



$_3[8\,|\,9,11]_7$



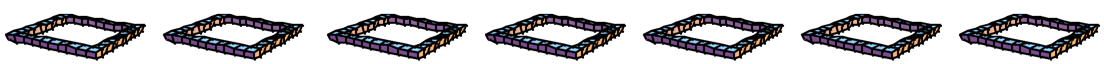
$S_3[8\,|\,9,11]_7$

Next, 4-rings of the same ringwidth are collected and stacked in one of three tiers. Finally the three tiers are stacked to form the SKYSCRAPER $S_3[8\,|\,9,11]_7$ (above right).

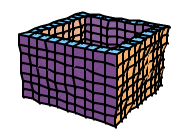
A TOWER $T_q[n\,|\,n+2,n+2]_s$ may be regarded either as

(a) a stack of the s one-story ZIGGURATS of the regular ZIGGURAT COMPLEX $_q[n\,|\,n+2,n+2]_s$ or

(b) a one-tier regular SKYSCRAPER. For the TOWER $T_1[8\,|\,10,10]_7$ (below right), q=1, s= r =7, and w=a=b=10.



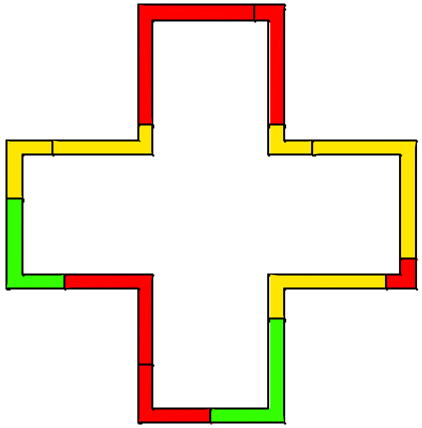
$_1[8\,|\,10,10]_7$



$S_1[8\,|\,10,10]_7$

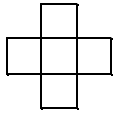
25. Cruciform 12-rings, Toltec diamonds, and Toltec TOWERS

The cruciform 12-ring depicted at the right is a natural extrapolation from a 4-ring. We will call such a 12-ring *uniform*, because the twelve straight segments of which it is composed are all of the same length. By analogy with the ringwidth of a 4-ring, we define the ringwidth of a uniform 12-ring as the sum of the lengths of the two contiguous arms of each pair of adjacent LOMINOES in the ring. The eighty-four LOMINOES of three L8 sets can be arranged to tile seven uniform 12-rings of ringwidth 10, which when stacked form a 12-ring TOWER that we call a *Toltec* TOWER.

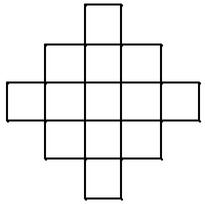


A 12-ring of ringwidth 10 (n=8)

Let us define a *Toltec diamond* of order 1 to be a unit square, and a Toltec diamond of order n to be an assembly of $2n(n-1)+1$ unit squares arranged as a stack of $2n$-1 centered rows of squares, the k^{th} row having length min($2k$-1,$4n$-$2k$-1) [EKLP 1992]. Toltec diamonds of order 2 and 3 are shown at the right. The *spine* of a uniform 12-ring, which is a circuit composed of the longitudinal midlines of its twelve segments, has the same shape as the external boundary of a Toltec diamond of order 2. The spine of a uniform 20-ring has the same shape as the external boundary of a Toltec diamond of order 3 (*etc.*). It is conjectured that five L8 sets tile seven uniform 20-rings of ringwidth 10, seven L8 sets tile nine uniform 28-rings of ringwidth 10 (*etc.*).



A Toltec diamond of order 2

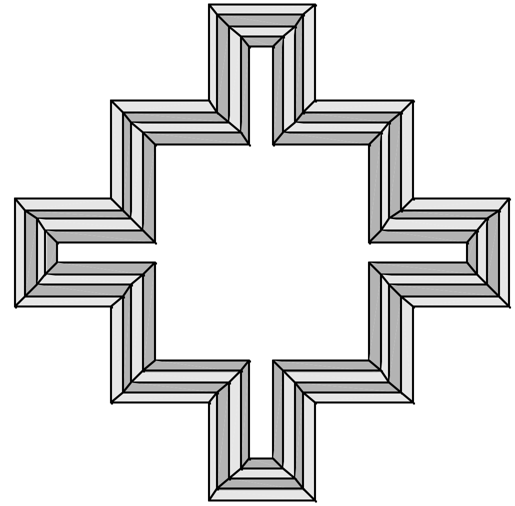


A Toltec diamond of order 3

26. Toltec rings and a L8 Toltec ZIGGURAT

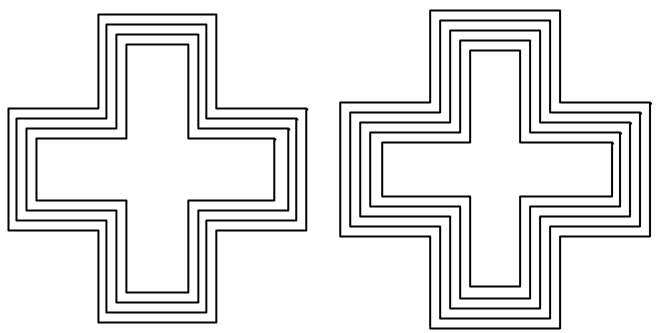
Let us call a 12-ring, 20-ring (*etc.*) a *Toltec ring*. A Toltec ring of ringwidth n+2 tiled by LOMINOES of Ln is an example of a uniform Toltec ring (*cf.* p. 43).

The outermost Toltec ring in the nested cluster shown at the right is a uniform 20-ring of ringwidth ten. The three innermost rings are *non-uniform* 20-rings; each of them is composed of sixteen *long segments* of length 10 and four *short segments* of even length<10. It is convenient to define the ringwidth of a non-uniform Toltec ring as *the length of each of its short segments*. The three innermost Toltec rings in the cluster shown at the right have ringwidths 4, 6, and 8.



Four nested Toltec 20-rings
The outermost 20-ring is uniform
and of ringwidth 10.
The three inner 20-rings
are non-uniform,
and their ringwidths are 8, 6, and 4.

A tiling has been found for the seven 12-rings, of ringwidths 7 to 13, of a Toltec ZIGGURAT composed of the LOMINOES of three L8 sets. The ring structure of this ZIGGURAT is illustrated below.



12-rings of ringwidths 8, 10, and 12 (left)
and 12-rings of ringwidths 7, 9, 11, and 13 (right)
of a L8 Toltec ZIGGURAT

27. The volume $V_{\mathbf{rings}}(a,b)$ of a set of $\boldsymbol{r}$ 4-rings of consecutive ringwidths from a to b

Let $V_{\text{rings}}(a,b)$=the volume of $\mathfrak{R}_{a,b}$, a consecutive set of $\boldsymbol{r}$ 4-rings of ringwidths a, a+1, a+2, …, b. Then

$$\boxed{V_{\text{rings}}(a,b)=2(\boldsymbol{b}-\boldsymbol{a}+1)(\boldsymbol{a}+\boldsymbol{b})} \tag{27.1}$$

where $\boldsymbol{a}=a-1$ and $\boldsymbol{b}=b-1$.

Proof:

The volume of a single 4-ring of ringwidth w is

$$v(w)=4(w-1). \tag{27.2}$$

The volume of $\boldsymbol{r}$ 4-rings of average ringwidth w_{Av} is therefore

$$\text{vol}=4\boldsymbol{r}(w_{\text{Av}}-1). \tag{27.3}$$

But

$$w_{\text{Av}}=(a+b)/2. \tag{27.4}$$

Hence

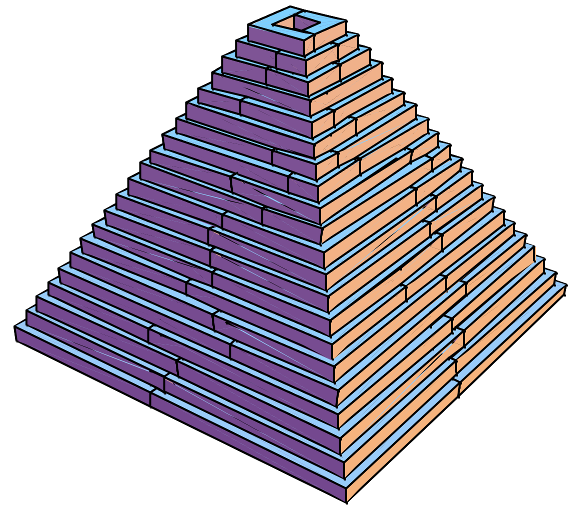
$$\begin{aligned} V_{\text{rings}}(a,b)&=\text{vol}\\ &=4\boldsymbol{r}\,[(a+b)/2-1]\\ &=2\boldsymbol{r}\,[a+b-2]. \end{aligned} \tag{27.5}$$

If we substitute in Eq. 27.5 for $\boldsymbol{a}=a-1$, $\boldsymbol{b}=b-1$, and $\boldsymbol{r}=\boldsymbol{b}-\boldsymbol{a}+1$, we obtain

$$V_{\text{rings}}(a,b)=2(\boldsymbol{b}-\boldsymbol{a}+1)(\boldsymbol{a}+\boldsymbol{b}). \tag{27.6}$$

□

28. A hypothetical solitary ZIGGURAT that cannot be packed



$_1[13|4{,}24]_1$
A solitary ZIGGURAT
for which no packing exists

The 21-story $_1[13|4{,}24]_1$, appears at first sight to be a plausible solitary ZIGGURAT candidate of irregular standard type. The volume $V_{\mathrm{set}}(13)=1092$ (*cf.* p. 31), which must be equal to the combined ring volume $V_{\mathrm{rings}}(a,b)=2(\mathbf{b}-\mathbf{a}+1)(\mathbf{a}+\mathbf{b})$ (*cf.* p. 45), with $(a,b)=(4,24)$, $\mathbf{a}=a-1$, and $\mathbf{b}=b-1$. For $n=13$, $a=4$ and $b=24$ are the smallest and largest possible ringwidths for top and bottom *z*-rings, respectively (*cf.* p. 47). To prove that no packing exists, it is sufficient to prove that the three largest *z*-rings, whose volumes are equal to 92, 88, and 84, cannot be constructed simultaneously from L13.

Summary of proof:

Each of the three largest *z*-rings must be a 4-ring, since it is impossible to construct either 2-rings or 3-rings of ringwidth greater than 13 from the pieces of L13 (*cf.* Inequality 3-1). An exhaustive search proves that L13 allows only the following two signatures for the bottom 4-ring of ringwidth 24:

$S_1(24)=<[12,13][11,13][11,12][12,12]>$ and $S_2(24)=<[13,13][11,12][12,13][11,11]>$.

Moreover of the fifteen possible signatures for a 4-ring of ringwidth 23, it is found that only

$S_1(23)=<[13,13][10,11][12,10][13,10]>$ has no pieces in common with $S_1(24)$, and only

$S_2(23)=<[12,12][11,13][10,13][10,11]>$ has no pieces in common with $S_2(24)$.

But after choosing either $S_1(24)$ and $S_1(23)$ or $S_2(24)$ and $S_2(23)$, it is impossible to find four pieces among the remaining seventy-two that can form the third largest 4-ring, which is of ringwidth 22 and volume 84. (Among these seventy-two remaining pieces, there is no set of four that have combined volume greater than 82.) Hence no packing exists. □

29. Why are there no examples of solitary standard ZIGGURATS (type 1:1) for n>11?

Let us call a set $\mathfrak{R}_{a,b}$ of z-rings of ringwidths a, a+1,..., b-1, b a *consecutive set* of z-rings.

Let $a_{\min}$ denote the smallest possible value for a in a consecutive set of at least two z-rings. To prove that $a_{\min}$ cannot be less than four, it is sufficient to observe that Ln cannot simultaneously tile a z-ring of ringwidth 3 and a z-ring of ringwidth 4, since the only possible z-ring of ringwidth 3 is the 2-ring with signature <[2,2][1,0][3,3][0,1]>, and every tiling of a z-ring of ringwidth 4 requires either [2,2] or [3,3]. □

It is similarly easy to prove that the largest possible ringwidth of a z-ring is 2n-2. (A 4-ring with signature <[n,n][n-2,n-1][n-1,n][n-2,n-2]>, for example, has ringwidth equal to 2n-2.) The volume of a 4-ring of ringwidth 2n-1 is 8n-8, but the combined volume of the four largest **LOMINOES** in Ln is only 8n-9. □

Let us call a consecutive set of z-rings *maximal* if a=4 and b=2n-2. We have just proved (above) that

No consecutive set of z-rings of volume greater than that of a maximal set can be tiled by the LOMINOES of Ln**.**

Suppose a=4 and b=2n-2. Then $\boldsymbol{a}$ =a-1=3 and $\boldsymbol{b}$ =b-1=2n-3. The volume of a maximal set $\mathfrak{R}_{a,b}$ (*cf.* Eq. 27.1) is

$$\begin{aligned} V_{\text{rings}}(a,b) &= 2(\boldsymbol{b} - \boldsymbol{a}\ 1)(\boldsymbol{a} + \boldsymbol{b}) \\ &= 4n(2n-5). \end{aligned} \tag{29.1}$$

We now prove that

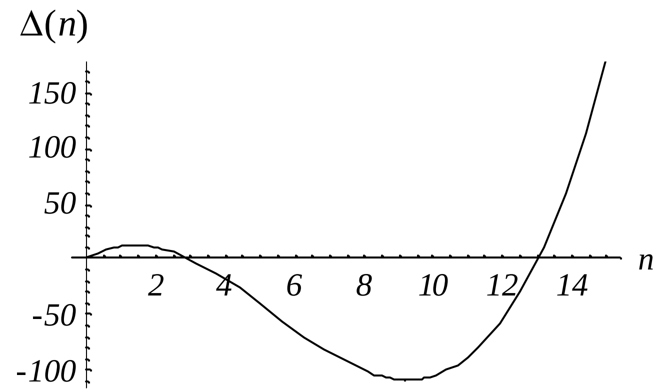
(a) for n=13, the volume $V_{\text{set}}(13)$ of L13 is exactly equal to $4n(2n-5)$, and
(b) for n>13, the volume $V_{\text{set}}(n)$ of Ln is greater than $4n(2n-5)$, and
a solitary standard ZIGGURAT for n>13 is therefore impossible.

To prove (a), we set n=13 in the expressions $4n(2n-5)$ and $V_{\text{set}}(n)=n(n^2-1)/2$ (*cf.* Eq. 15.3), obtaining the value 1092 in both cases. □

To prove (b), we define

$$\Delta(n) = V_{\text{set}}(n) - V_{\text{rings}}(4,2n-2)$$

$$= n(n-3)(n-13)/2. \tag{29.2}$$



The only zeroes of $\Delta(n)$ are 0, 3, and 13 (*cf.* graph at right). Since there are no zeroes of $\Delta(n)$ for $n>13$, and $\Delta(n)$ is positive for any arbitrarily chosen $n>13$, we conclude that $\Delta(n)$ is positive for *all* $n>13$, *i.e.*,

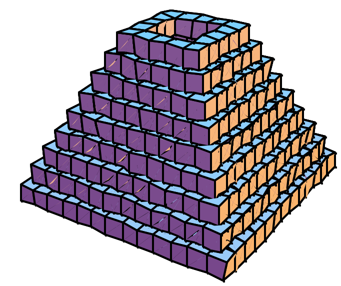
$$V_{\text{set}}(n) > V_{\text{rings}}(4,2n-2) \text{ for all } n>13. \qquad \square$$

By considering every possible consecutive set of *z*-rings for $n<13$, we have also proved that for $n<13$, there are no standard ZIGGURATS of type 1:1, either regular or irregular, aside from the five examples listed on p. 34. □

Summary: *For $n>11$ there are no examples – either regular or irregular – of solitary standard ZIGGURATS (type 1:1). For $n>13$, the volume $V(n)=n(n^2-1)/2$ of the standard set Ln $(=O(n^3))$ is greater than the volume $V_{rings}(4,2n-2)=4n(2n-5)$ $(=O(n^2))$ of $\mathfrak{R}_{a,b}$, the maximal set of z-rings.*

30. A second solitary ZIGGURAT for which no packing exists

Consider the hypothetical irregular solitary augmented ZIGGURAT ${}_1[8^\dagger|5{,}13]_1$ (right). Because fifty-nine packings were found, in a computer search, for the somewhat similar irregular solitary *standard* ZIGGURAT ${}_1[8|4{,}12]_1$, it seemed plausible that packings of ${}_1[8^\dagger|5{,}13]_1$ might also exist. However, after thirty-one of the thirty-two pieces were placed, in each of several attempts (using physical pieces) to find a packing, the single remaining piece was always found to be of the wrong shape! An exhaustive computer search then demonstrated that no packing exists. For *z*-rings of ringwidth ≤ 8, 2-ring, 3-ring, and 4-ring compositions were included. First a count was made of the total number of ways *z*-rings of ringwidths 5 and 6 can be tiled with no pieces in common. Then similar counts were made for *z*-rings of every ringwidth from 5 to 7, 5 to 8, …, 5 to 13. The search failed spectacularly at the last stage, as shown below.

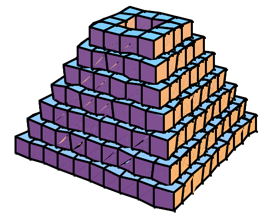


${}_1[8^\dagger|5{,}13]_1$
A solitary ZIGGURAT
for which no packing exists

ringwidths of *z*-rings	number of ways of simultaneously tiling the *z*-rings whose ringwidths are listed in the column at left
5	5
5	5
5,6	12
5,6,7	325
5,6,7,8	5343
5,6,7,8,9	52821
5,6,7,8,9,10	808728
5,6,7,8,9,10,11	2522033
5,6,7,8,9,10,11,12	994201
5,6,7,8,9,10,11,12,13	0

31. A third solitary ZIGGURAT for which no packing exists

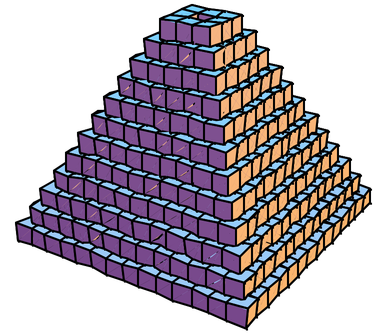
The hypothetical irregular standard solitary ZIGGURAT ${}_1[7\,|\,4{,}10]_1$, composed of the twenty-one pieces of L7, is shown at the right. Like ${}_1[8^\dagger\,|\,5{,}13]_1$ (*cf.* p. 49), this example *almost* admits a packing, *i.e.,* all except one of the pieces can be properly placed. An exhaustive computer search has demonstrated, however, that no packing exists. For the *z*-rings of ringwidth ≤ 7, 2-ring, 3-ring, and 4-ring compositions were included. First a count was made of the total number of ways *z*-rings of ringwidths 4 and 5 can be tiled simultaneously (*i.e.,* with no pieces in common). Then similar counts were made successively for *z*-rings of ringwidths 4 to 6, 4 to 7, …, 4 to 10. Just as in the search for ${}_1[8^\dagger\,|\,5{,}13]_1$, this search failed at the last stage, as shown below.



${}_1[7\,|\,4{,}10]_1$
A solitary ZIGGURAT
for which no packing exists

ringwidths of *z*-rings	number of ways of simultaneously tiling the *z*-rings whose ringwidths are listed in the column at left
4	2
4,5	4
4,5,6	13
4,5,6,7	67
4,5,6,7,8	33
4,5,6,7,8,9,	9
4,5,6,7,8,9,10	0

32. Additional examples of ZIGGURATS that cannot be packed



$_1[9\,|\,3,14]_1$
A singular ZIGGURAT
for which no packing exists

The possibility of packing the irregular standard solitary ZIGGURAT $_1[9\,|\,3,14]_1$ (right) can be ruled out by an argument that was cited on p. 47:

No standard set Ln can tile both a z-ring of ringwidth 3 and a z-ring of ringwidth 4. Hence the smallest possible z-ring in a ZIGGURAT packed by Ln is of ringwidth 4. □

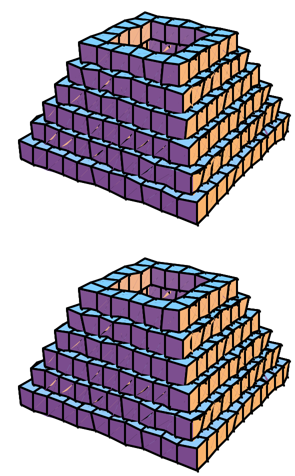
The impossibility of packing the irregular standard ZIGGURAT COMPLEX $_1[9\,|\,6,11]_2$ (below right) may be deduced as follows:

> Because there are thirty-six LOMINOES in L9, $z_{Av}=3$ for the z-rings of $_1[9\,|\,6,11]_2$. This implies that either
>
> (a) every one of the twelve z-rings is a 3-ring, or else
> (b) the number of 2-rings is equal to the number of 4-rings.

But (a) is ruled out, because the two z-rings of ringwidth 10 and the two z-rings of ringwidth 11 all require four LOMINOES each. It then follows from (b) that the COMPLEX must contain at least four 2-rings. However, no more than two 2-rings can be tiled with the pieces of L9. For proof, consider the only possible signature for a 2-ring composed of the pieces of L9:

$$<[i,\ i]\ [1,\ 0]\ [i+1,\ i+1]\ [0,\ 1]> \quad (2 \le i \le 8).$$

Since $a=6$, each of the four required 2-rings needs two LOMINOES of ringwidth $w \ge 5$ from subset 1 of L9. But L9 contains a total of only five such pieces. Therefore only two, not four, 2-rings can be tiled. Hence $_1[9\,|\,6,11]_2$ cannot be packed. □



$_1[9\,|\,6,11]_2$
A ZIGGURAT COMPLEX
for which no packing exists

33. A periodic table for ZIGGURATS (period=8)

The volume of Ln is $V_{\text{set}}(n)=n(n^2\text{-}1)/2=\mathrm{O}(n^3)$, but $V_{\text{rings}}(4,2n\text{-}2)$, the volume of a maximal consecutive set $\mathfrak{R}_{4,2n\text{-}2}$ of z-rings, is equal to $4n(2n\text{-}5)=\mathrm{O}(n^2)$ (*cf.* p. 47).
Let $n=8k+j$. If $j=0$ or 1, $s=k$; if $j=2$ or 3, $s=4k+1$; if $j=4$ or 5, $s=2k+1$; if $j=6$ or 7, $s=4k+3$.

q = number of sets				s = number of ZIGGURATS			
4	L2	sets	have volume equal to that of	1	1-story	ZIGGURAT	$_4[2\,\vert\,4,4]_1$
4	L3	sets	have volume equal to that of	1	3-story	ZIGGURAT	$_4[3\,\vert\,4,6]_1$
2	L4	sets	have volume equal to that of	1	3-story	ZIGGURAT	$_2[4\,\vert\,5,7]_1$
2	L5	sets	have volume equal to that of	1	5-story	ZIGGURAT	$_2[5\,\vert\,5,9]_1$
4	L6	sets	have volume equal to that of	3	5-story	ZIGGURATS	$_4[6\,\vert\,6,10]_3$
4	L7	sets	have volume equal to that of	3	7-story	ZIGGURATS	$_4[7\,\vert\,6,12]_3$
1	L8	set	has volume equal to that of	1	7-story	ZIGGURAT	$_1[8\,\vert\,7,13]_1$
1	L9	set	has volume equal to that of	1	9-story	ZIGGURAT	$_1[9\,\vert\,7,15]_1$
4	L10	sets	have volume equal to that of	5	9-story	ZIGGURATS	$_4[10\,\vert\,8,16]_5$
4	L11	sets	have volume equal to that of	5	11-story	ZIGGURATS	$_4[11\,\vert\,8,18]_5$
2	L12	sets	have volume equal to that of	3	11-story	ZIGGURATS	$_2[12\,\vert\,9,19]_3$
2	L13	sets	have volume equal to that of	3	13-story	ZIGGURATS	$_2[13\,\vert\,9,21]_3$
4	L14	sets	have volume equal to that of	7	13-story	ZIGGURATS	$_4[14\,\vert\,10,22]_7$
4	L15	sets	have volume equal to that of	7	15-story	ZIGGURATS	$_4[15\,\vert\,10,24]_7$
1	L16	set	has volume equal to that of	2	15-story	ZIGGURATS	$_1[16\,\vert\,11,25]_2$
1	L17	set	has volume equal to that of	2	17-story	ZIGGURATS	$_1[17\,\vert\,11,27]_2$
4	L18	sets	have volume equal to that of	9	17-story	ZIGGURATS	$_4[18\,\vert\,12,28]_9$
4	L19	sets	have volume equal to that of	9	19-story	ZIGGURATS	$_4[19\,\vert\,12,30]_9$
2	L20	sets	have volume equal to that of	5	19-story	ZIGGURATS	$_2[20\,\vert\,13,31]_5$
2	L21	sets	have volume equal to that of	5	21-story	ZIGGURATS	$_2[21\,\vert\,13,33]_5$
4	L22	sets	have volume equal to that of	11	21-story	ZIGGURATS	$_4[22\,\vert\,14,34]_{11}$
4	L23	sets	have volume equal to that of	11	23-story	ZIGGURATS	$_4[23\,\vert\,14,36]_{11}$
1	L24	set	has volume equal to that of	3	23-story	ZIGGURATS	$_1[24\,\vert\,15,37]_3$
1	L25	set	has volume equal to that of	3	25-story	ZIGGURATS	$_1[25\,\vert\,15,39]_3$

34. The n+2 regular standard ZIGGURAT COMPLEXES for $2 \leq n \leq 9$

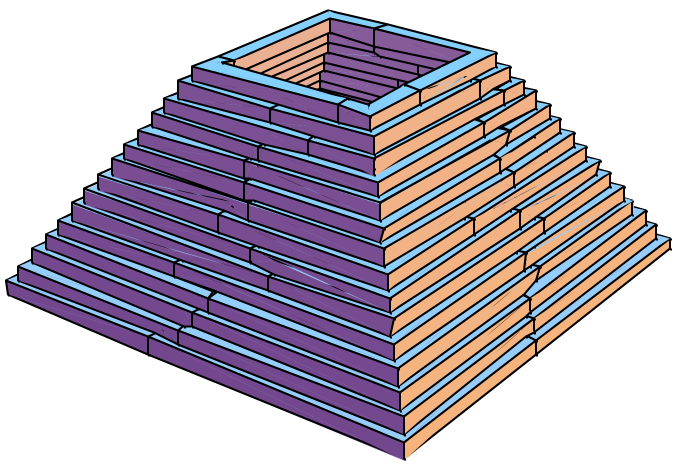
For every $n \geq 2$, there are n-1 examples for which the smallest ringwidth $a \geq 4$. The entries for $a \leq 3$ are fictitious, since no packing by Ln is possible for $a<4$ (*cf.* p. 47). **Boldface** entries describe ZIGGURAT COMPLEXES for which both q and s have minimum possible values. These boldface examples are the first eight listed on p. 52.

n	$V(n)$	$N(n)$	a	b	r	q	s
2	3	1	**4**	**4**	**1**	**4**	**1**
			3	5	3	12	1
			2	6	5	20	1
			1	7	7	28	1
3	12	3	5	5	1	4	3
			4	**6**	**3**	**4**	**1**
			3	7	5	20	3
			2	8	7	28	3
			1	9	9	12	1
4	30	6	6	6	1	2	3
			5	**7**	**3**	**2**	**1**
			4	8	5	10	3
			3	9	7	14	3
			2	10	9	6	1
			1	11	11	22	3
5	60	10	7	7	1	2	5
			6	8	3	6	5
			5	**9**	**5**	**2**	**1**
			4	10	7	14	5
			3	11	9	18	5
			2	12	11	22	5
			1	13	13	26	5
6	105	15	8	8	1	4	15
			7	9	3	4	5
			6	**10**	**5**	**4**	**3**
			5	11	7	28	15
			4	12	9	12	5
			3	13	11	44	15
			2	14	13	52	15
			1	15	15	4	1
7	168	21	9	9	1	4	21
			8	10	3	4	7
			7	11	5	20	21
			6	**12**	**7**	**4**	**3**
			5	13	9	12	7
			4	14	11	44	21
			3	15	13	52	21
			2	16	15	20	7
			1	17	17	68	21
8	252	28	10	10	1	1	7
			9	11	3	3	7
			8	12	5	5	7
			7	**13**	**7**	**1**	**1**
			6	14	9	9	7
			5	15	11	11	7
			4	16	13	13	7
			3	17	15	15	7
			2	8	17	17	7
			1	19	19	19	7
9	360	36	11	11	1	1	9
			10	12	3	1	3
			9	13	5	5	9
			8	14	7	7	9
			7	**15**	**9**	**1**	**1**
			6	16	11	11	9
			5	17	13	13	9
			4	18	15	5	3
			3	19	17	17	9
			2	20	19	19	9
			1	21	21	7	3

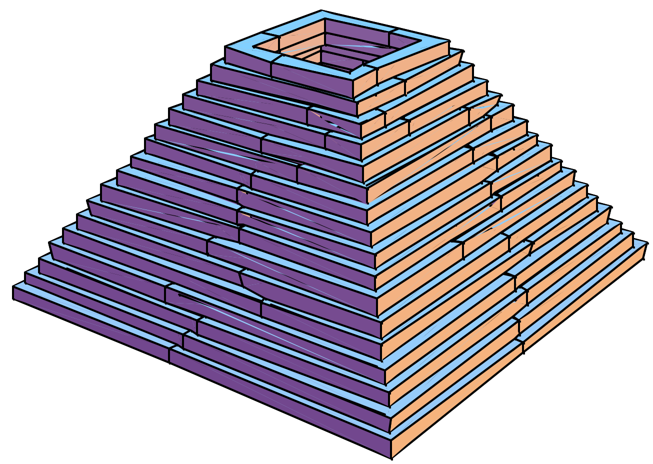
35. *Medial* ZIGGURAT COMPLEXES

We denote the regular standard ZIGGURAT COMPLEXES $MZ_{8k}={}_1[8k\,|\,4k+3,12k+1]_k$ and $MZ_{8k+1}={}_1[8k+1\,|\,4k+3,12k+3]_k$ (k=1,2,…) as *medial ZIGGURAT COMPLEXES.* The next larger examples following the two solitary ZIGGURATS $MZ_8={}_1[8\,|\,7,13]_1$ (*cf.* p. 30) and $MZ_9={}_1[9\,|\,7,15]_1$ (*cf.* p. 34) are $MZ_{16}={}_1[16\,|\,11,25]_2$ and $MZ_{17}={}_1[17\,|\,11,27]_2$.

The irregular ZIGGURAT COMPLEX ${}_1[16\,|\,8,24]_2$ is related to ${}_1[16\,|\,11,25]_2$ in the same way the irregular ${}_1[8\,|\,4,12]_1$ is related to ${}_1[8\,|\,7,13]_1$: the bottom 4-ring of ${}_1[16\,|\,11,25]_2$ is absent in ${}_1[16\,|\,8,24]_2$ and instead there are three additional z-rings at the top of ${}_1[16\,|\,8,24]_2$. The transformation of ${}_1[8k\,|\,4k+3,12k+1]_k$ into ${}_1[8k\,|\,4k,12k]_k$ applies to every MZ_{8k}, since $12k=(4k-1)+4k+(4k+1)$ for $k\geq 1$. ${}_1[16\,|\,11,25]_2$ and ${}_1[16\,|\,8,24]_2$ are shown below (in fictitious packings). It is conjectured that every MZ_{8k} and every ${}_1[8k\,|\,4k,12k]_k$ can be packed by L8k and also that every MZ_{8k+1} can be packed by L($8k$+1).



One of the two congruent *regular* ZIGGURATS
of the conjectured medial
ZIGGURAT COMPLEX ${}_1[16\,|\,11,25]_2$



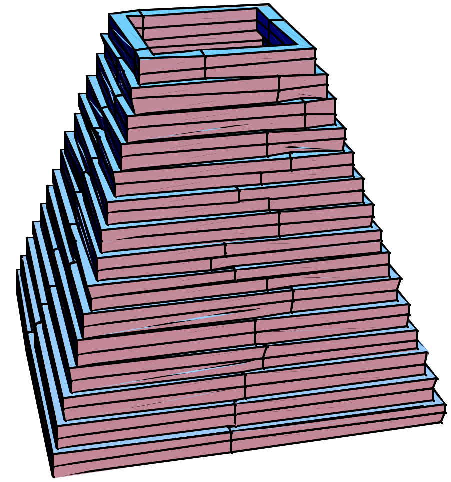
One of the two congruent *irregular* ZIGGURATS
of the conjectured medial
ZIGGURAT COMPLEX ${}_1[16\,|\,8,24]_2$

36. Truncation indices of ZIGGURATS and SKYSCRAPERS

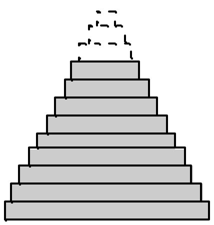
The SKYSCRAPER $S_1[16\,|\,8,24]_2$ is shown in two views at the right.

The ZIGGURAT ${}_q[n\,|\,a,b]_s$ is composed of $r = b-a+1$ z-rings. It may be transformed into a non-truncated stepped pyramid if it is capped by a small stepped pyramid Π with bottom ring of ringwidth a-1 and top ring consisting of a single cube (ringwidth=1). The number of floors of Π is equal to a-1. Two examples of such pyramid-capped ZIGGURATS are shown in outline below.

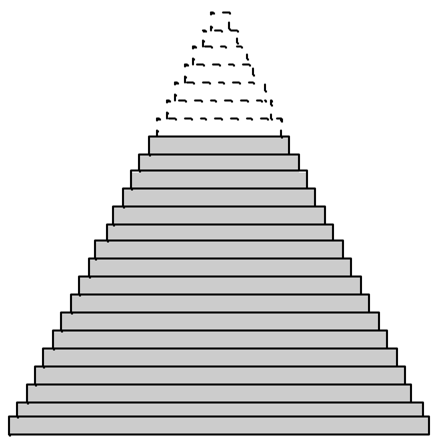
We define the **truncation index** g of each of the ZIGGURATS in $S_q[n\,|\,a,b]_s$ as $r\,/\,b$. The truncation index is a measure of the degree to which a ZIGGURAT approximates a stepped pyramid. For medial ZIGGURATS (*cf.* p. 54), the asymptotic value of g, *i.e.*, $\lim_{k\to\infty} g\,(k)$, is 2/3. For the SKYSCRAPER $S_q[n\,|\,a,b]_s$, we define the truncation index as $g_S = s\,r\,/\,b$.



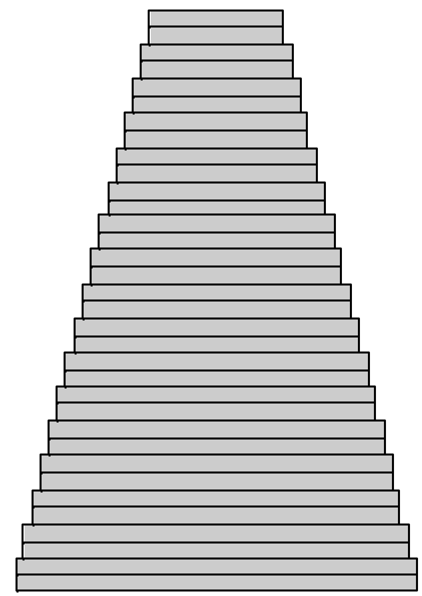
$S_1[16\,|\,8,24]_2$



${}_1[8\,|\,4,12]_1$
with a three-story
pyramidal cap



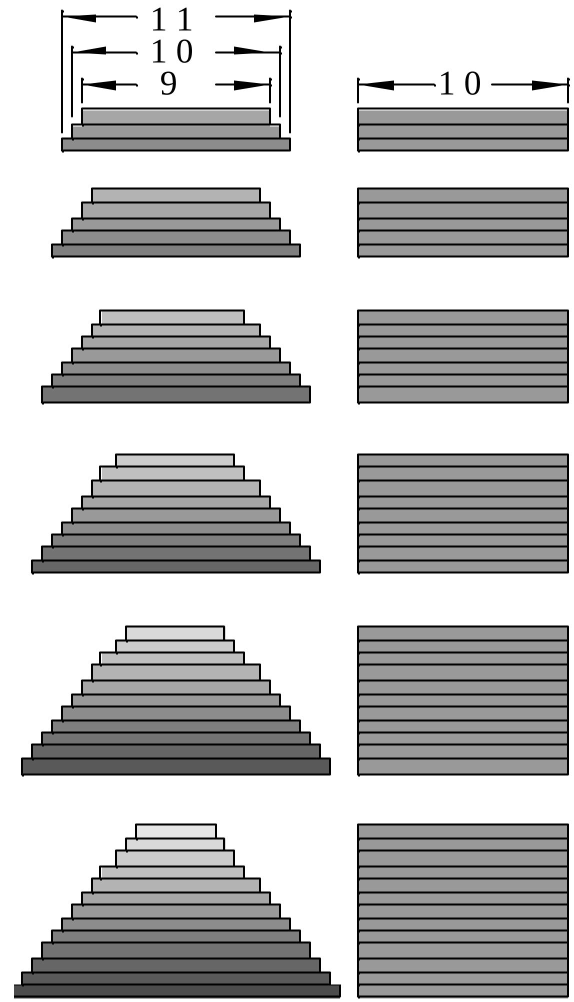
${}_1[16\,|\,8,24]_2$
with a seven-story
pyramidal cap



$S_1[16\,|\,8,24]_2$

37. Volumes of ZIGGURAT COMPLEXES and TOWERS

The central 4-ring of the regular standard ZIGGURAT COMPLEX ${}_{q}[n\,|\,a,b]_{s}$ has ringwidth n+2. The remaining r-1 4-rings can be associated in pairs, each of which consists of (a) a 4-ring of ringwidth n+2-k lying k stories above the central 4-ring and (b) a 4-ring of ringwidth n+2+k lying k stories below the central 4-ring ($1 \le k \le (r-1)/2$). The combined volume of each such pair is equal to the volume of two 4-rings of the TOWER $T_{q}[n\,|\,n+2,n+2]_{s}$. At the right are outlines of r-story ZIGGURATS ($3 \le r \le 13$; odd r) from the ZIGGURAT COMPLEXES ${}_{3}[8|9,11]_{7}$, ${}_{5}[8|8,12]_{7}$, ${}_{1}[8|7,13]_{1}$, ${}_{9}[8|6,14]_{7}$, ${}_{11}[8|5,15]_{7}$, and ${}_{13}[8|4,16]_{7}$ (*cf.* p. 53). Next to each ZIGGURAT is the outline of the corresponding TOWER, which is a stack of r 4-rings of ringwidth 10.



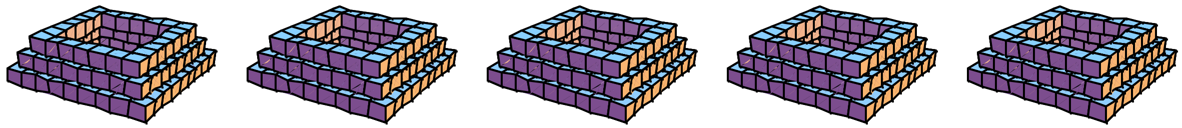
ZIGGURATS (left)
and TOWERS (right)
of equal volume

The number of ways of choosing four LOMINOES to tile 4-rings of various ringwidths is discussed on pp. 61-64. Using trial-and-error, it is enormously harder to find a packing for a ZIGGURAT or ZIGGURAT COMPLEX than for the corresponding TOWER, especially if $r \gg 1$. For n>5, it is found that the number of packing solutions is considerably greater for a TOWER than for the associated ZIGGURAT. For example, the solitary ZIGGURAT ${}_{1}[8|7,13]_{1}$ has 384 solutions, but even an incomplete sampling of packings of the TOWER $T_1[8|10,10]_7$ yielded 4426 solutions. For the ZIGGURAT ${}_{1}[9|7,15]_{1}$, the total number of packings is 6772. A sparse sampling of packings of the corresponding TOWER $T_1[9|11,11]_9$ yielded only 2060 solutions, but it is conjectured that the total number is very much larger.

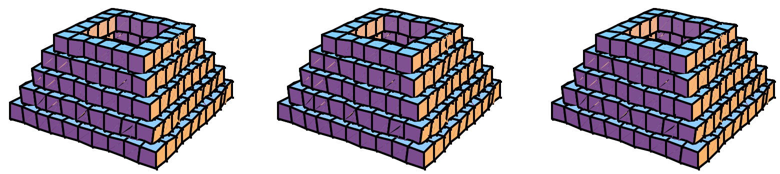
A computer search has revealed that there are exactly 800 packings of the *pied* TOWER $T_1[8^{\dagger}|10,10]_8$, in which every 4-ring is composed of four differently colored LOMINOES (*cf.* p. 3).

38. Regular standard SKYSCRAPERS composed of four sets of L6

The pieces of *four* L6 sets can be arranged to form either the *five* ZIGGURATS of the ZIGGURAT COMPLEX $_4[6\,|\,7,9]_5$ or the *three* ZIGGURATS of the ZIGGURAT COMPLEX $_4[6\,|\,6,10]_3$.



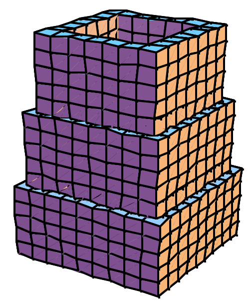
$_4[6\,|\,7,9]_5$



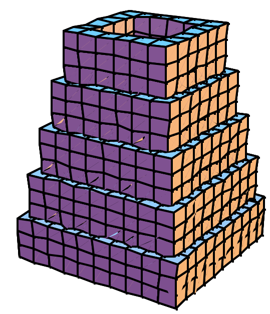
$_4[6\,|\,6,10]_3$

The three-tier SKYSCRAPER $S_4[6\,|\,7,9]_5$ (left) is formed by combining 4-rings of the same ringwidth in $_4[6\,|\,7,9]_5$ in each of three separate tiers of height five and then stacking the three tiers.

The five-tier SKYSCRAPER $S_4[6\,|\,6,10]_3$ (right) is formed by combining 4-rings of the same ringwidth in $_4[6\,|\,6,10]_3$ in each of five separate tiers of height three and then stacking the five tiers.



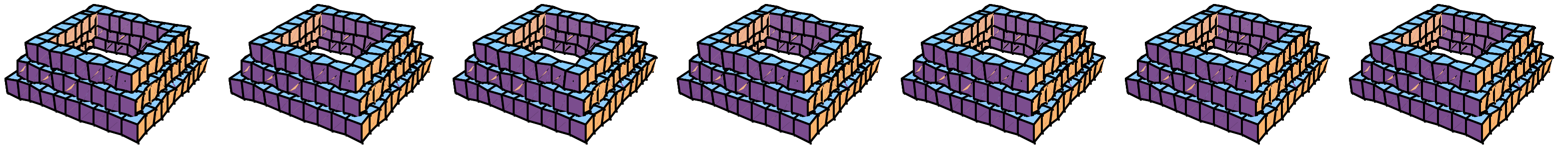
$S_4[6|7,9]_5$



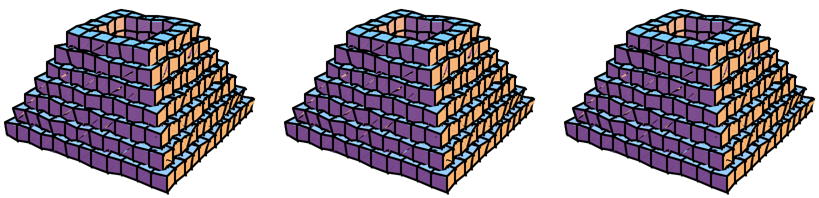
$S_4[6|6,10]_3$

39. Regular standard SKYSCRAPERS composed of four sets of L7

The pieces of *four* L7 sets can be arranged to form either the *seven* ZIGGURATS of the ZIGGURAT COMPLEX $_4[7|8,10]_7$ or the *three* ZIGGURATS of the ZIGGURAT COMPLEX $_4[7|6,12]_3$.



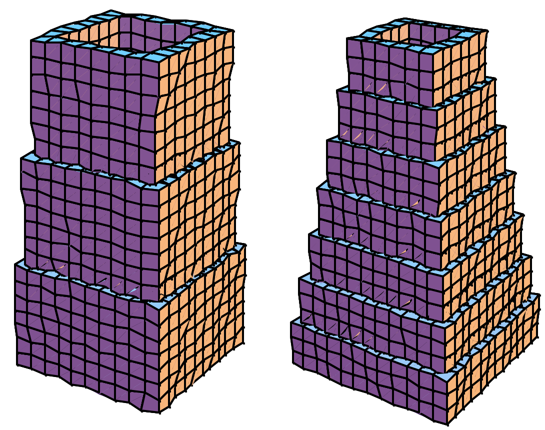
$_4[7|8,10]_7$



$_4[7|6,12]_3$

The three-tier SKYSCRAPER $S_4[7|8,10]_7$ (left) is formed by combining 4-rings of the same ringwidth in $_4[7|8,10]_7$ in each of three separate tiers of height seven and then stacking the three tiers.

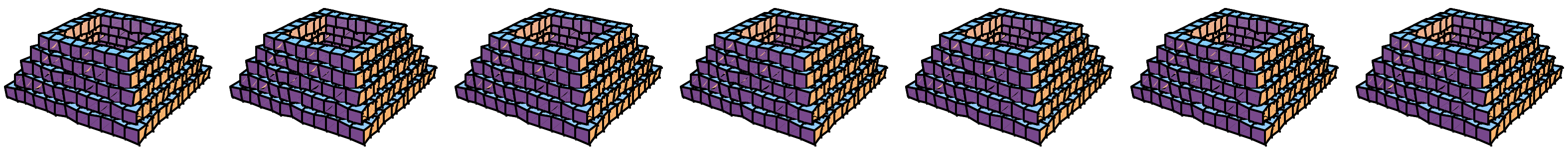
The seven-tier SKYSCRAPER $S_4[7|6,12]_3$ (right) is formed by combining 4-rings of the same ringwidth in $_4[7|6,12]_3$ in each of seven separate tiers of height three and then stacking the seven tiers.



$S_4[7|8,10]_7$ $S_4[7|6,12]_3$

40. Regular standard SKYSCRAPER composed of five sets of L8

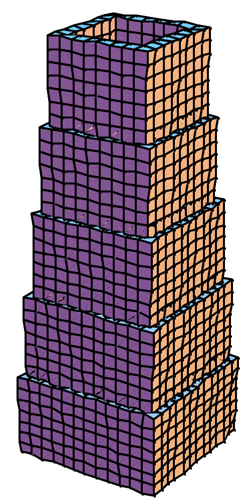
The pieces of *five* L8 sets can be arranged to form the *seven* ZIGGURATS of the ZIGGURAT COMPLEX $_5[8|8{,}12]_7$.



$_5[8|8{,}12]_7$

The five-tier SKYSCRAPER $S_5[8|8{,}12]_7$ (right) is formed by combining 4-rings of the same ringwidth in $_5[8|8{,}12]_7$ in each of five separate tiers of height seven and then stacking the five tiers.

The three-tier SKYSCRAPER $S_3[8|9{,}11]_7$ is described on p. 42.



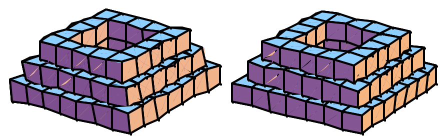
$S_5[8|8{,}12]_7$

41. Examples of regular augmented ZIGGURAT COMPLEXES

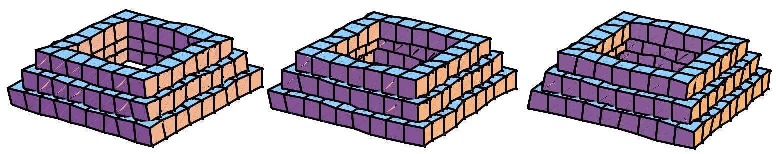
There are no solitary regular augmented ZIGGURATS, but below are shown two examples of regular augmented ZIGGURAT COMPLEXES that are of relatively simple type:

$${}_3[4^\dagger\,|\,5,7]_2 \text{ (type 3:2) and } {}_2[6^\dagger\,|\,7,9]_3 \text{ (type 2:3).}$$

Packing solutions for these two examples are not hard to find.



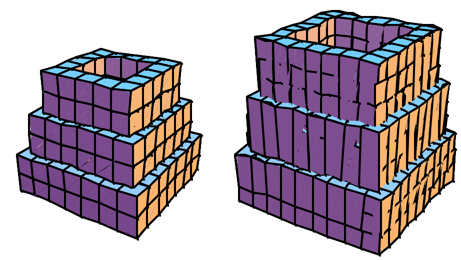
${}_3[4^\dagger\,|\,5,7]_2$



${}_2[6^\dagger\,|\,7,9]_3$.

The three-tier SKYSCRAPER $S_3[4^\dagger\,|\,5,7]_2$ (left) is formed by combining 4-rings of the same ringwidth in ${}_3[4^\dagger\,|\,5,7]_2$ in each of three separate tiers of height two and then stacking the three tiers.

The three-tier SKYSCRAPER $S_2[6^\dagger\,|\,7,9]_3$ (right) is formed by combining 4-rings of the same ringwidth in ${}_2[6^\dagger\,|\,7,9]_3$ in each of three separate tiers of height three and then stacking the three tiers.



$S_3[4^\dagger\,|\,5,7]_2$ $S_2[6^\dagger\,|\,7,9]_3$

42. Enumerating the combinations of LOMINOES that tile a consecutive set $\mathfrak{R}_{a,b}$ of 4-rings

The full tree search algorithm employed in a computer program to find tilings of a r-story singular regular standard ZIGGURAT $_q[n\,|\,a,b]_1$ performs the following tasks in sequence:

(1) identify every four-piece set S_w that tiles a 4-ring of ringwidth w ($a \le w \le b$);

(2) compare every instance of S_a with every instance of S_{a+1} and identify every *disjoint pair* $\{S_a, S_{a+1}\}$, *i.e.*, every pair of sets S_a and S_{a+1} that contain eight distinct LOMINOES;

(3) compare every instance of the *disjoint pair* $\{S_a, S_{a+1}\}$ with every instance of S_{a+2} and identify every *disjoint triple* $\{S_a, S_{a+1}, S_{a+2}\}$;

...

(4) compare every instance of a *disjoint (r-1)-tuple* $\{S_a, S_{a+1}, \ldots, S_{b-1}\}$ with every instance of S_b and identify every *disjoint r-tuple* $P(a,b)=\{S_a, S_{a+1}, \ldots, S_{b-1}, S_b\}$. Every instance of $P(a,b)$ defines a packing of $_q[n\,|\,a,b]_1$.

The above algorithm employs a divide-and-conquer strategy. The comparisons are performed in r-1 separate steps, thereby avoiding the combinatorial explosion that would result if all the instances of every set S_w were compared in a single step. For $n \approx 4$ or 5, however, a single-step algorithm is quite feasible.

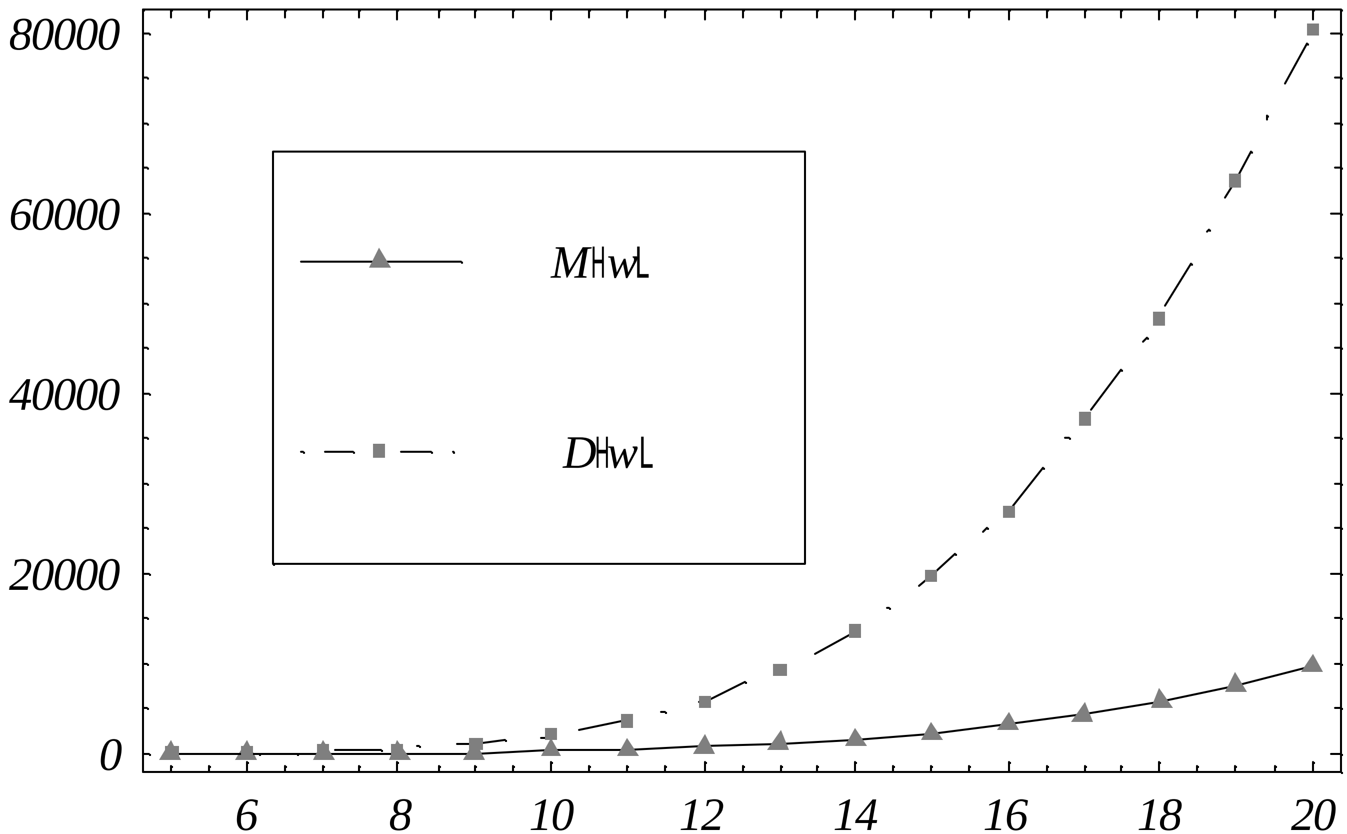
In the initial nested-loop enumeration of the four-piece sets S_w for each value of w (step (1) above), eight instances – not one instance – of each set S_w are counted, because of the cyclic and anti-cyclic permutations of pieces inherent in the loop structure. Furthermore, 4-rings with the same composition but different arrangements are distinguished initially (*cf.* p. 4). These duplications are eliminated after each count is performed. For each value of $w \le 20$, the number $M(w)$ of sets in the reduced list is smaller by a factor between 8 and 9 than the number $D(w)$ that includes duplicates.

For $5 \le w \le 20$, polynomials of degree four with rational coefficients provide exact values for both $M(w)$ and $D(w)$ (*cf.* p. 63). It is conjectured that these results hold also for $w > 20$.

43. $D(w)$ **and** $M(w)$ **for a standard LOMINOES set** Ln **of order** $n \geq w\text{-}2$ **(**$5 \leq w \leq 20$**)**

w	5	6	7	8	9	10	11	12
$D(w)$	0	16	168	408	1080	1944	3696	577
$M(w)$	0	2	15	44	111	216	402	656

w	13	14	15	16	17	18	19	20
$D(w)$	9360	13440	19800	26856	37128	48328	63840	80544
$M(w)$	1050	1550	2265	3132	4305	5684	7476	9536



44. Polynomial expressions for $D(w)$ and $M(w)$

For 4-rings of ringwidth $w \geq 5$, we call $M(w)$ the *4-ring multiplicity*. $M(w)$ is of greater interest than $D(w)$, because it doesn't include duplicates (*cf.* p. 61).

Empirical results for $M(w)$ and $D(w)$ obtained by a computer search for ZIGGURAT packings (*cf.* p. 61) suggest the following conjectures:

---

odd $w \geq 5$: $M(w) = (w-3)(w-5)(w^2-5w+1)/8.$ (44.1)

even $w \geq 6$: $M(w) = (w-4)^2(w^2-5w-2)/8.$ (44.2)

---

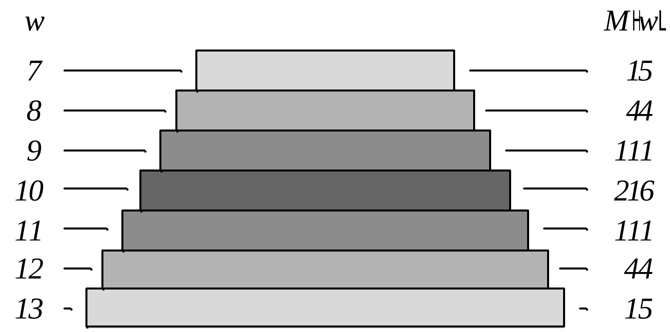
odd $w \geq 5$: $D(w) = w(w-3)(w-4)(w-5).$ (44.3)

even $w \geq 6$: $D(w) = (w-4)(w^3-8w^2+11w+14).$ (44.4)

(*Cf.* pp. 92-97 for further discussion of multiplicity.)

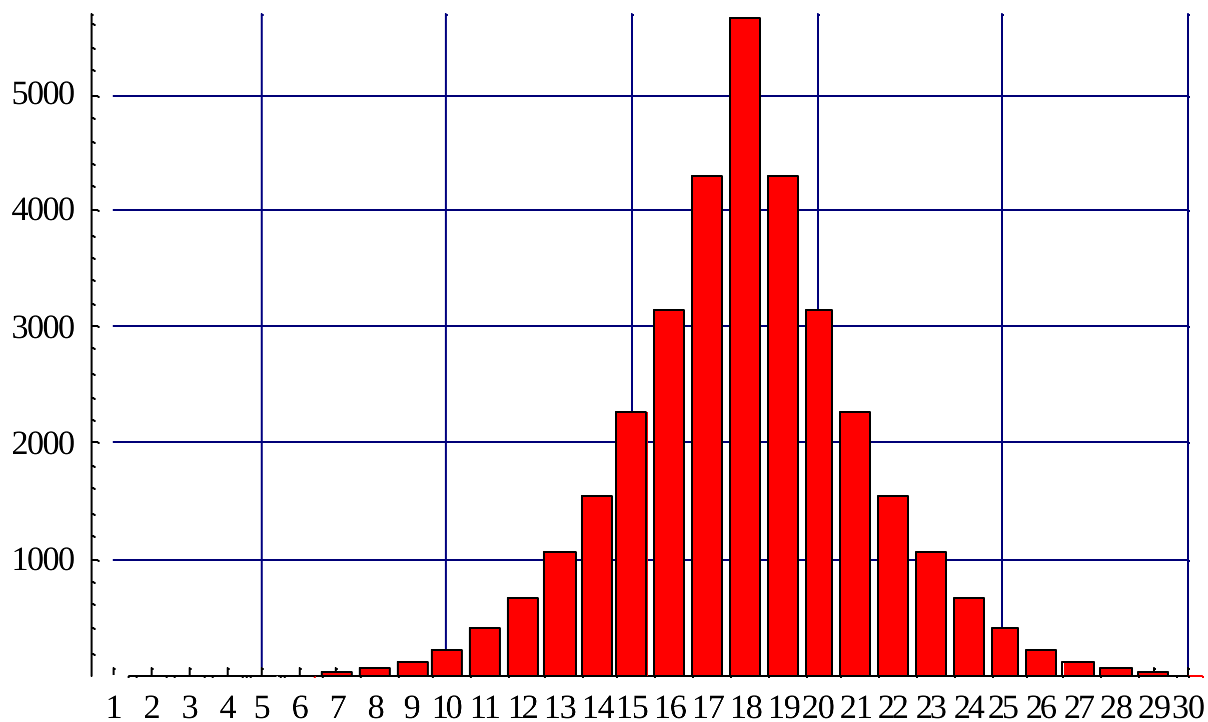
45. Duality and the bilateral symmetry of $M(w)$ with respect to $w=n+2$

The conjectured polynomial expressions for $M(w)$ and $D(w)$ (*cf.* Eqs. 44.1 to 44.4) apply to 4-rings of ringwidth $5 \le w \le 2n-2$. As a consequence of the dual relation for pairs of LOMINOES pieces (*cf.* p. 5), $M(w)=M(2n+4-w)$. At the right are values of $M(w)$ for the seven 4-rings of $_1[8\,|\,7,13]_1$.



$M(w)$ *vs.* w for 4-rings assembled from the pieces of L16

Plotted at right is the multiplicity $M(w)$ *vs.* w for 4-rings assembled from pieces of the standard set L16 (*cf.* p. 62). For every standard set Ln, the maximum multiplicity occurs at $w=n+2$. (This is hardly surprising, since 4-rings of ringwidth $n+2$ are the only ones in which *every* piece of Ln can fit).



46. The number of 4-rings in every solitary regular standard ZIGGURAT is *odd*.

Proof:

Every solitary regular standard ZIGGURAT ${}_1[n\,|\,a,b]_1$ is comprised of a consecutive set $\mathfrak{R}_{a,b}$ of $\mathbf{r}$ 4-rings of ringwidths a, $a+1$, $a+2$, …, b, where

$$\mathbf{r}=b-a+1. \tag{46.1}$$

Let $\mathbf{w}=w-1$, where w is the ringwidth of a 4-ring. The volume of a 4 ring is equal to $4\mathbf{w}$. Let $\mathbf{a}=a-1$ and $\mathbf{b}=b-1$. On p. 45 we proved that the volume of the $\mathbf{r}$ 4-rings in $\mathfrak{R}_{a,b}$ is

$$V_{rings}(a,b)=2(\mathbf{b}-\mathbf{a}+1)(\mathbf{a}+\mathbf{b}). \tag{46.2}$$

The combined volume of the $N(n)=n(n-1)/2$ LOMINOES in Ln is (*cf.* Eq. 15.3 on p. 31)

$$V_{set}(n)=n(n^2-1)/2. \tag{46.3}$$

Since $V_{rings}(a,b)=V_{set}(n)$, by combining Eqs. 46.2 and 46.3 we obtain

$$n(n^2-1)/2=2(\mathbf{b}-\mathbf{a}+1)(\mathbf{a}+\mathbf{b}). \tag{46.4}$$

Since there are four LOMINOES in every 4-ring, the number $N(n)$ of LOMINOES in Ln is four times the number $\mathbf{r}$ of 4-rings in ${}_1[n\,|\,a,b]_1$.

$$\begin{aligned} n(n-1)/2 &= 4(b-a+1) \\ &= 4(\mathbf{b}-\mathbf{a}+1). \end{aligned} \tag{46.5}$$

Substituting Eq. 46.5 in Eq. 46.4 gives

$$2(n+1) = a + b\ . \tag{46.6}$$

Since

$$r = b - a + 1$$

$$= 2b - (a + b) + 1, \tag{46.7}$$

if we substitute from Eq. 46.6 for $a + b$ in Eq. 46.7, we obtain

$$r = 2b - 2(n+1) + 1$$

$$= 2(b - n) - 1. \tag{46.8}$$

Hence r is odd.

Next we prove that r is odd also for the ZIGGURATS of every ZIGGURAT COMPLEX.

47. The number of 4-rings in each ZIGGURAT of a regular standard ZIGGURAT COMPLEX is *odd*.

Let $\mathbf{r}$ =the number of 4-rings in each of the ZIGGURATS $Z_{a,b}$ in the set of s ZIGGURATS that comprise the regular standard ZIGGURAT COMPLEX ${}_q[n\,|\,a,b]_s$. **Then $\mathbf{r}$ is odd.**

Proof:

${}_q[n\,|\,a,b]_s$ contains s copies of the consecutive set $\mathfrak{R}_{a,b}$ of $\mathbf{r}$ 4-rings of ringwidths a, a+1, a+2, …, b, where

$$\mathbf{r} = b\text{-}a+1. \tag{47.1}$$

Let $\mathbf{a} = a$-1 and $\mathbf{b} = b$-1. The number $4s(\mathbf{b}-\mathbf{a}+1)$ of LOMINOES in s copies of $\mathfrak{R}_{a,b}$ is equal to the number $qn(n\text{-}1)/2$ of LOMINOES in q copies of Ln:

$$qn(n\text{-}1)/2 = 4s(\mathbf{b}-\mathbf{a}+1). \tag{47.2}$$

The total volume of q copies of Ln is

$$qn(n^2\text{-}1)/2, \tag{47.3}$$

which is equal to the volume $2s(\mathbf{a}+\mathbf{b})(\mathbf{b}-\mathbf{a}+1)$ of s copies of $\mathfrak{R}_{a,b}$.

Hence

$$qn(n^2\text{-}1)/2 = 2s(\mathbf{b}-\mathbf{a}+1)(\mathbf{a}+\mathbf{b}), \tag{47.4}$$

and – substituting Eq. 47.2 in Eq. 47.4 – we find that

$$2(n+1) = \mathbf{a}+\mathbf{b}. \tag{47.5}$$

After making the same substitutions as in Eqs. 46.7 and 46.8, we conclude that $\mathbf{r}$ is odd. □

48. Nested square ANNULI tiled by LOMINOES sets Ln, L(n+8), L(n+16), …

If we treat LOMINOES as flat tiles used to tile finite plane regions, it is convenient to ignore the thickness of the tile and define the area of each piece as the area of its top surface. Since all the pieces are of unit thickness, the area of each piece is numerically equal to its volume. The area of the standard set Ln is denoted as $A(n)=n(n^2\text{-}1)/2$. Let us define the positive integers $u(n)$ and $v(n)$ as follows:

$$u(n)=n^2-1+8n, \tag{48.1}$$

$$v(n)=|\,n^2-1-8n\,|\,. \tag{48.2}$$

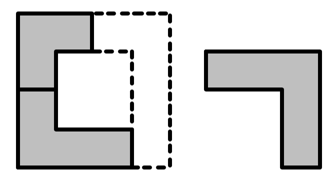
Then

$$A(n)=[u(n)^2 - v(n)^2]/64. \tag{48.3}$$

For odd $n\geq 3$, it follows from Eqs. 48.1-48.3 that $A(n)$ is equal to the area of a square ANNULUS T_n whose outer and inner boundaries are concentric squares with integer edge lengths $u(n)/8$ and $v(n)/8$, respectively. It is conjectured that

$\boxed{\text{for odd } n\geq 5, \text{ L}n \text{ tiles the corresponding ANNULUS } T_n.}$

The ANNULUS T_3 cannot be tiled by the three LOMINOES of L3, as is demonstrated at the right.



T_3 and the three pieces of L3

Eqs. 48.1 and 48.2 imply that for $n\geq 1$, $v(n+8)=u(n)$. Let us define the following four families of snugly nested ANNULI:

$$\text{F}_1=\{T_9, T_{17}, T_{25}, \ldots\},\ \text{F}_3=\{T_{11}, T_{19}, T_{27}, \ldots\},\ \text{F}_5=\{T_5, T_{13}, T_{21}, \ldots\},\ \text{F}_7=\{T_7, T_{15}, T_{23}, \ldots\}.$$

We conjecture that single-set tilings by Ln exist for every annulus T_n in each of these sets. Each family of ANNULI tiles the entire plane, except for a single square hole at the center of the tiling. On pp. 69-70, tilings of T_n by Ln are shown for n=5, 7, 9, 11, and 13.

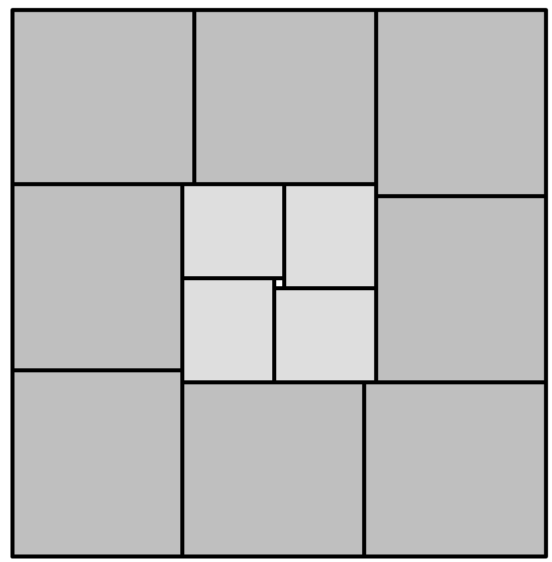
49. Lengths of the inner and outer edges of the ANNULI of the families F_1, F_3, F_5, and F_7

$u(n)/8$=edge length of the outer square
$v(n)/8$=edge length of the inner square

F_1			F_3			F_5			F_7		
n	$u(n)/8$	$v(n)/8$	n	$u(n)/8$	$v(n)/8$	n	$u(n)/8$	$v(n)/8$	n	$u(n)/8$	$v(n)/8$
9	19	1	3	4	2	5	8	2	7	13	1
17	53	19	11	26	4	13	34	8	15	43	13
25	103	53	19	64	26	21	76	34	23	89	43
33	169	103	27	118	64	29	134	76	31	151	89
41	251	169	35	188	118	37	208	134	39	229	151

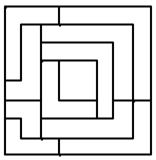
Every ANNULUS T_n of the family F_1 admits a trivial tiling by the corresponding Ln:

the $(n-1)/2$ pronic rectangles of Ln are concatenated to tile T_n (n=9, 17, 25, …), as illustrated at the right for T_9 (light gray) and T_{17} (dark gray). It appears that no such generic algorithm exists for tilings of any of the ANNULI of F_3, F_5, or F_7.

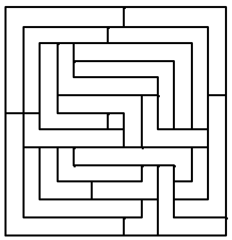


50. ANNULUS breadth

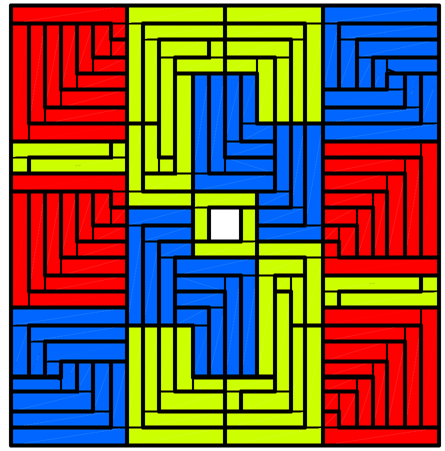
For odd $n \geq 9$, we define the *breadth* $\mathbf{d}$ of a one-set ANNULUS T_n as $\mathbf{d} = [u(n)-v(n)]/16$. It follows from Eqs. 48.1 and 48.2 that for odd $n \geq 9$, $v(n)=u(n)-16n$ and therefore that $\mathbf{d} = n$.



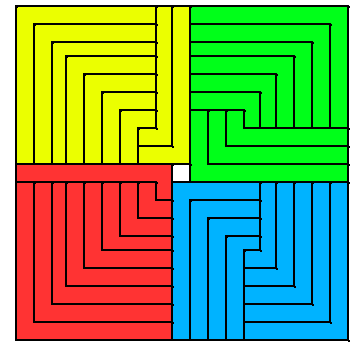
$8^2 - 2^2$
One L5 set



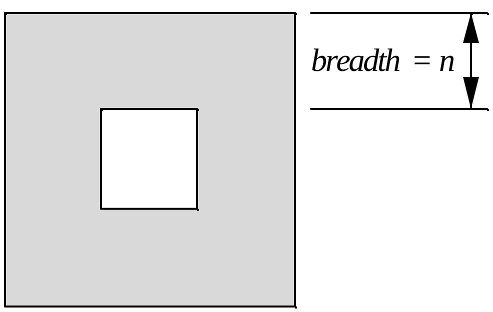
$13^2 - 1^2$
One L7 set



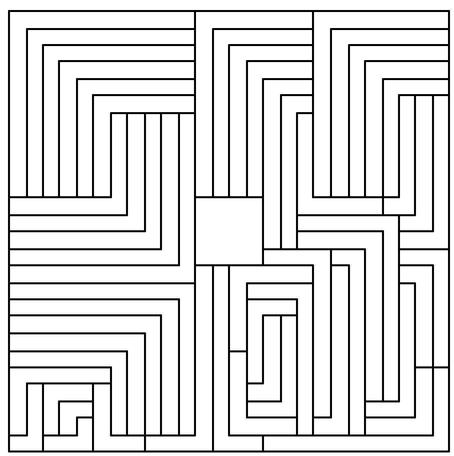
$26^2 - 2^2$
Four canonically colored L7 sets
(C2 symmetry)



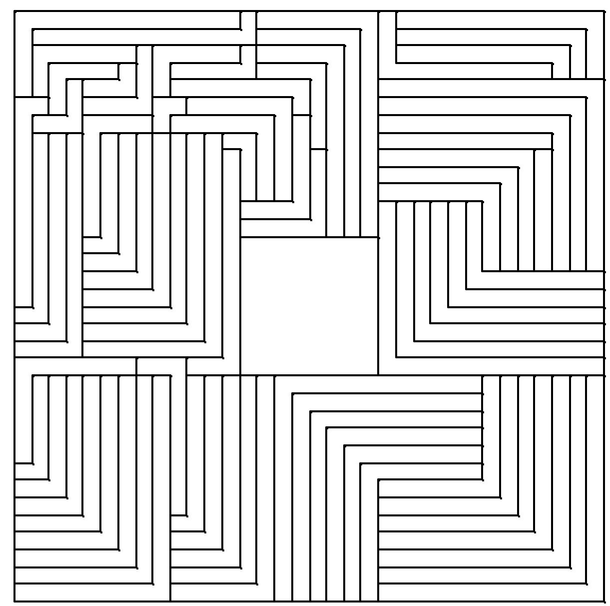
$19^2 - 1^2$
One L9 set



Annulus tiled by
one Ln set



$26^2 - 4^2$
One L11 set



$34^2 - 8^2$
One L13 set

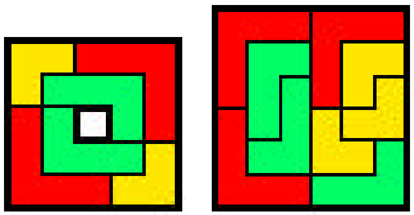
51. Recursive ANNULAR tilings

For $k \geq 3$, k sets of L3 tile the square ANNULUS $(k+3)^2 - (k-3)^2$.

The area $kA(3)$ of k sets of L3 is equal to $12k$, which is also the area of the ANNULUS $(k+3)^2 - (k-3)^2$. For $k \geq 4$, the breadth d of such an annulus (*cf.* p. 70) is $[(k+3)-(k-3)]/2=3$. The three LOMINOES of L3 tile one 3 x 4 pronic RECTANGLE, but the ANNULI $(k+3)^2 - (k-3)^2$ can be tiled by pronic rectangles only for $k \equiv 0 \pmod 4$. The ANNULI for $k=2$ and $k=3$ are shown below.

There are four families of ANNULI for L3:

Y_0: k=4, 8, 12, …($7^2 - 1^2$, $11^2 - 5^2$, $15^2 - 9^2$, …)
Y_1: k=5, 9, 13, …($8^2 - 2^2$, $12^2 - 6^2$, $16^2 - 10^2$, …)
Y_2: k=6, 10, 14, …($9^2 - 3^2$, $13^2 - 7^2$, $17^2 - 11^2$, …)
Y_3: k=7, 11, 15, …($10^2 - 4^2$, $14^2 - 8^2$, $18^2 - 12^2$, …)



$5^2 - (-1)^2$ (k=2) $6^2 - 0^2$ (k=3)

As illustrated on pp. 72-73, a simple recursive algorithm generates every ANNULUS of each of the four families listed above. The first and smallest ANNULUS in each family is denoted as its *nucleus*. The second ANNULUS in each family is obtained by inserting four 3 x 4 pronic rectangles into the nucleus. Every subsequent ANNULUS is similarly derived from its predecessor by the addition of four 3 x 4 pronic rectangles. In each of the illustrations on pp. 72-73, new pronic rectangles are shown in darker shades of gray to indicate that they have been added to the preceding ANNULUS.

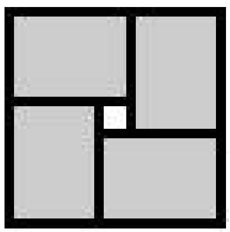
Y_1 and Y_3 define three different 'holey' tilings of the plane by concentric ANNULI chosen alternately from the two families.

$6^2 - 0^2$, $12^2 - 6^2$, $18^2 - 12^2$, $24^2 - 18^2$, …
$10^2 - 4^2$, $16^2 - 10^2$, $22^2 - 16^2$, $28^2 - 22^2$, …
$14^2 - 8^2$, $20^2 - 14^2$, $26^2 - 20^2$, $32^2 - 26^2$, …

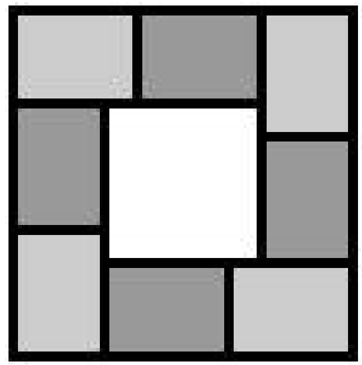
Y_0 and Y_2 define similar alternating tilings.

For what other values of n do k sets of Ln tile a square ANNULUS for all $k \geq k_{\min}$?

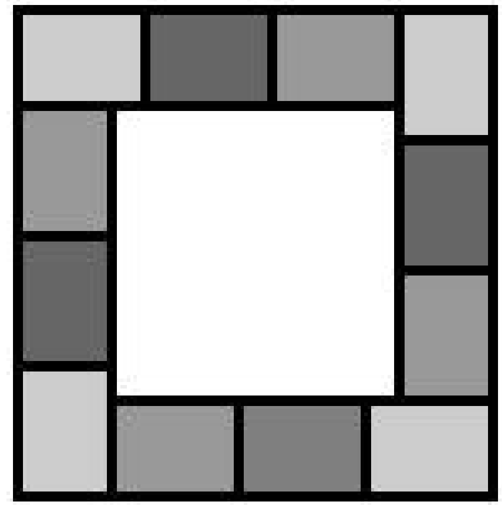
Y_0 and Y_1 – The first and second families of square ANNULI tiled by *k* sets of L3



$7^2 - 1^2$

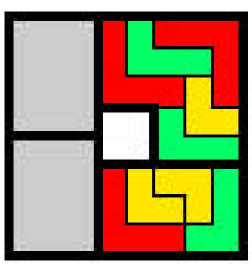


$11^2 – 5^2$

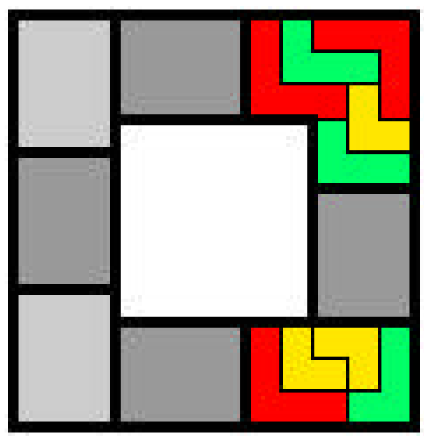


$15^2 - 9^2$

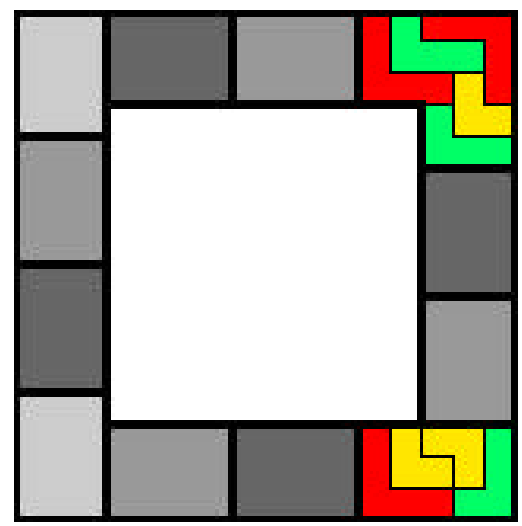
Y_0



$8^2 - 2^2$



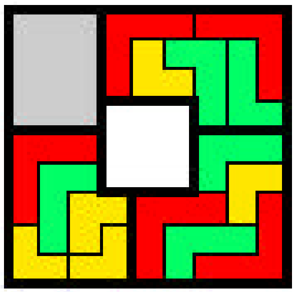
$12^2 - 6^2$



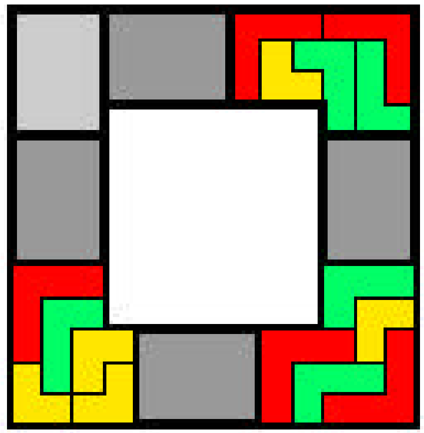
$16^2 - 10^2$

Y_1

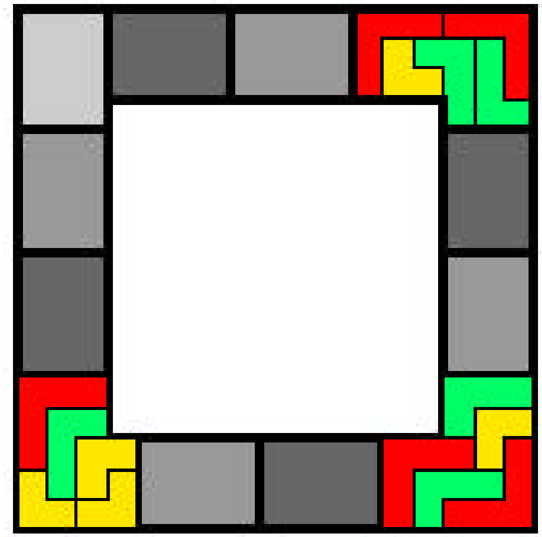
Y_2 and Y_3 – The third and fourth families of square ANNULI tiled by k sets of L3



$9^2 - 3^2$

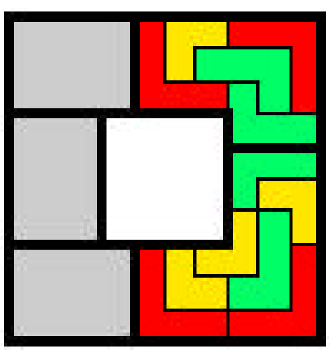


$13^2 - 7^2$

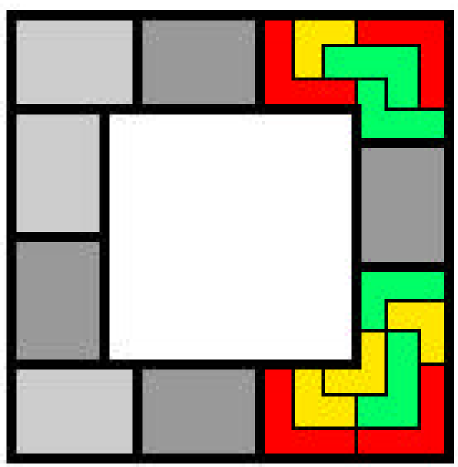


$17^2 - 11^2$

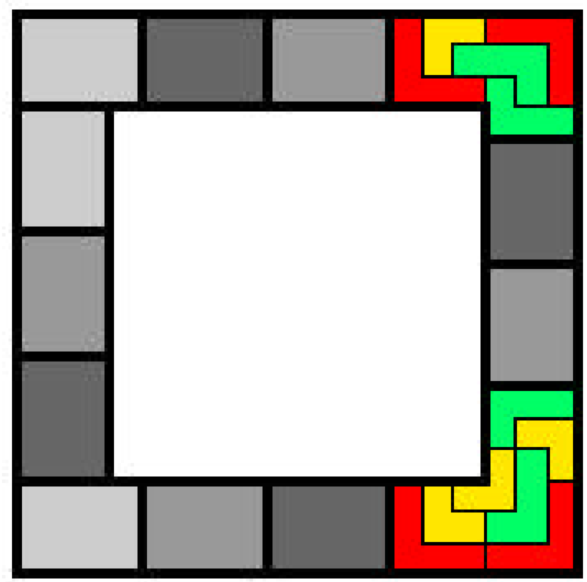
Y_2



$10^2 - 4^2$



$14^2 - 8^2$

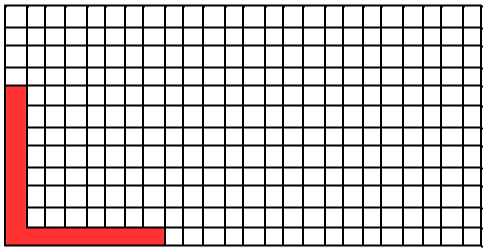


$18^2 - 12^2$

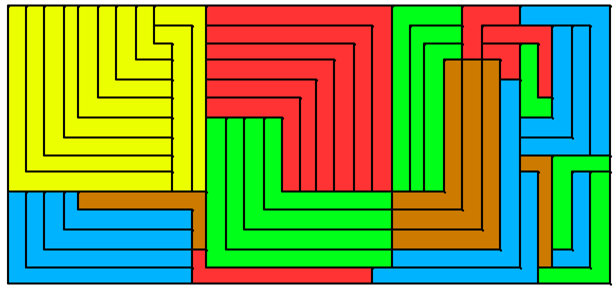
Y_3

52. Tiling rectangles and square ANNULI with standard or augmented sets of LOMINOES

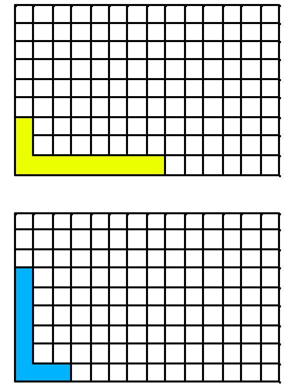
The area of a single standard set Ln is denoted as $A(n)$, and the area of a single augmented set L$n^\dagger$ is denoted as $A^\dagger(n)$. If we set $A(n)$ or $A^\dagger(n)=d_1d_2$ ($d_1 \geq n$ and $d_2 \geq n$), then candidates for u congruent rectangular arenas tiled by t sets of Ln or L$n^\dagger$ ($t,u \geq 1$) are defined by positive integer solutions of the equation t x (area of one set) = u x d_1d_2. The smallest rectangle tiled by one standard set is the 3 x 4 pronic rectangle tiled by L3. Tilings exist for each of the untiled arenas shown below and for a variety of other SQUARE or square ANNULAR arenas that you can easily discover yourself.



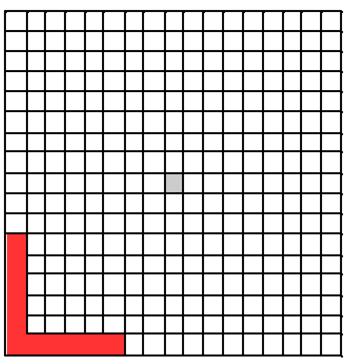
12 x 24 rectangle
One L8† set



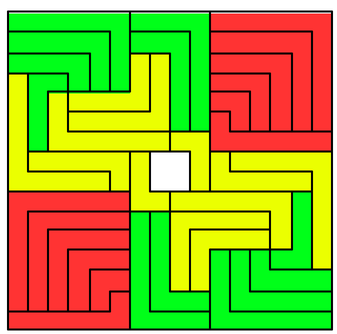
15 x 33 rectangle
One L10 set



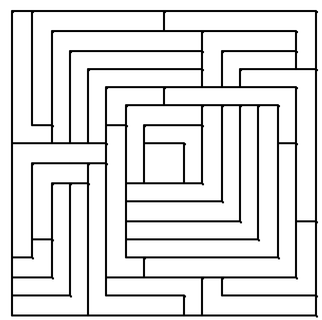
Two 9 x 14 rectangles
One L8 set



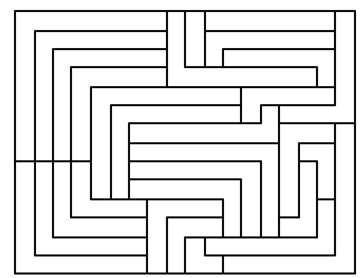
17^2 - 1^2 annulus
One L8† set



16^2 - 2^2 annulus
Two L6† sets



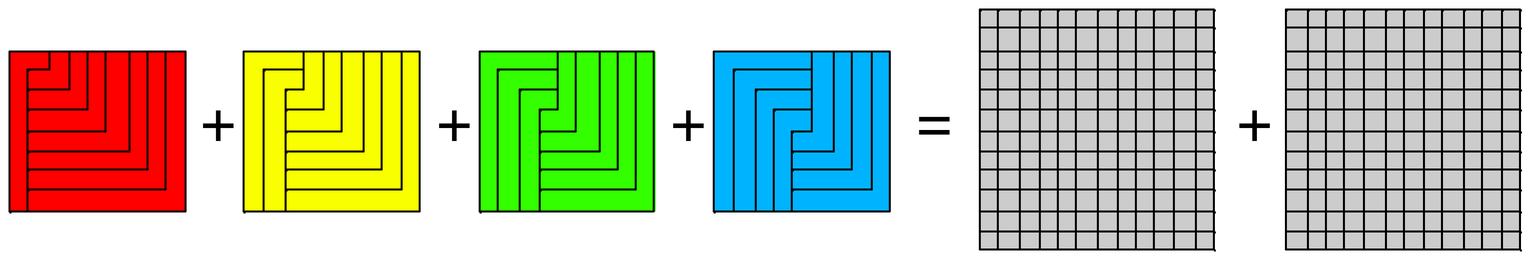
16^2 - 2^2 annulus
One L8 set



14 x 18 rectangle
One L8 set

53. Tiling two 12 x 12 SQUARES with one L8† set

The area $A^\dagger(n)$ of L$n^\dagger$ is $n^2(n+1)/2$. Hence $A^\dagger(8)=288=2 \times 12^2$. It is moderately challenging to find a tiling of two 12 x 12 SQUARES by the LOMINOES of one L8† set. Try it!



For $n<24$, there are no other examples of LOMINOES sets – either standard or augmented – whose area is equal to the sum of two or more equal squares (*cf.* p. 144). It has been proved that every positive integer can be expressed as the sum of four squares (one or more of which may be 0^2). Many such sums suggest possible tilings by LOMINOES sets, but the examples for $n \gg 8$ involve forbiddingly large numbers of pieces. Below are a few selected examples of sums of squares for $n<20$, including two – $A(15)$ and $A^\dagger(6)$ – for which the edge lengths of the squares define an arithmetic sequence.

$A(9) = 360 = 6^2+6^2+12^2+12^2$

$A(11) = 660 = 9^2+11^2+13^2+17^2$

$A(12) = 858 = 12^2+13^2+16^2+17^2$

$A(13) = 1092 = 16^2+16^2+16^2+18^2$

$A(15) = 1680 = 14^2+18^2+22^2+26^2$

$A(16) = 2040 = 10^2+12^2+14^2+40^2$

$A(16) = 2040 = 12^2+14^2+16^2+38^2$

$A(17) = 2448 = 18^2+18^2+30^2+30^2$

$A(17) = 2448 = 12^2+48^2$

$A^\dagger(10) = 550 = 10^2+15^2+15^2$

$A^\dagger(12) = 936 = 12^2+12^2+18^2+18^2$

$A^\dagger(16) = 2176 = 8^2+8^2+32^2+32^2$

$A^\dagger(18) = 3078 = 30^2+33^2+33^2$

Unfortunately there is no L6† tiling for

$A^\dagger(6) = 126 = 4^2+5^2+6^2+7^2$.

(Can you prove this?)

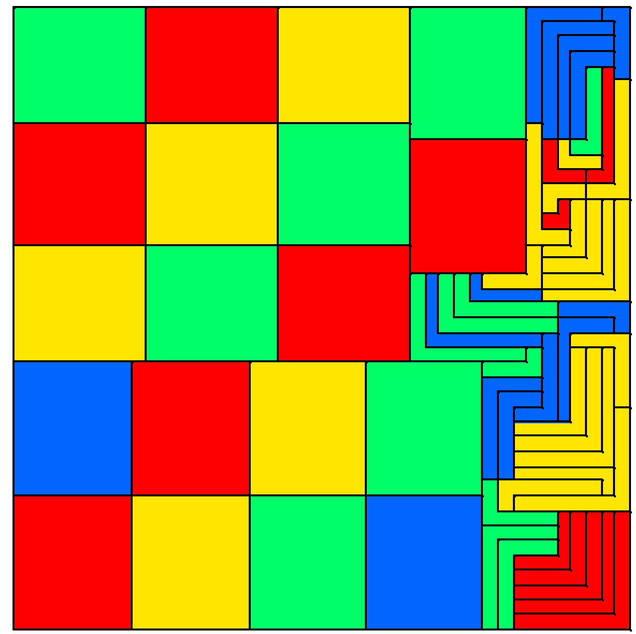
54. Tiling a SQUARE with Ln or L$n^\dagger$ sets

What is the smallest SQUARE that can be tiled by an integer number of either standard or augmented sets? Since $A(n)=n(n^2-1)/2$ and $A^\dagger(n)=n^2(n+1)/2$, we seek smallest integer solutions for (l, s) and (m, t) in Eqs. 54.1 and 54.2:

$$\text{L}n: \quad l\, n(n^2-1)/2 = s^2 \tag{54.1}$$

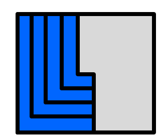
$$\text{L}n^\dagger: \quad m\, n^2(n+1)/2 = t^2 \tag{54.2}$$

For n=8, $A(8)$=252 and $A^\dagger(8)$=288. We find solutions (l, s)=(7,42) for L8 and (m, t)=(2,24) for L8†. Let us first discuss the 42 x 42 SQUARE, which can be tiled by seven L8 sets.



A tiling of the 42 x 42 square by seven L8 sets
Each colored rectangle is a 8 x 9 pronic rectangle.

The area of L8 is equal to the combined area of three and one-half 8 x 9 pronic rectangles (*cf.* p. 1). Subsets 1, 2, and 3 each tile one such rectangle. The half-subset 4 tiles one-half of a pronic rectangle (below right). We denote a LOMINO in a plane tiling as *embedded* if it is contained in a one-subset tiling of a n x $(n+1)$ pronic rectangle; otherwise it is called *dispersed*. It is a challenging problem to find a tiling of the 42 x 42 SQUARE by seven L8 sets in which the number of embedded pieces is as large as possible. In the tiling shown at the right, 152 of the 196 LOMINOES of seven L8 sets are embedded in nineteen pronic rectangles, while the remaining 44 LOMINOES are dispersed.



The half-subset 4

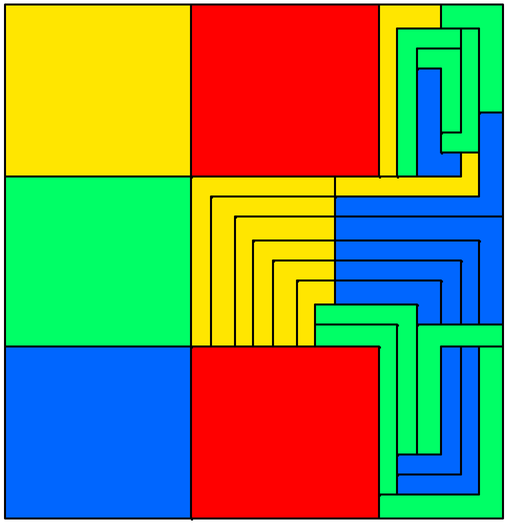
It is unknown whether there exists a tiling of the 42 x 42 SQUARE by seven L8 sets that contains more than 152 embedded pieces.

55. Tilings of SQUARES

Now consider the solution $(m,t)=(2,24)$ of Eq. 54.2 for the 24 x 24 SQUARE tiled by two L8† sets. Because it is *not* true that both eight and nine are divisors of twenty-four, a tiling in which all of the LOMINOES are embedded (*cf.* p. 76) is ruled out by the following theorem of David Klarner [HON 1976]:

The square $a \times a$ can be tiled with $b \times c$ rectangles if and only if both b and c divide a.

It is unknown whether there exists a tiling of the 24 x 24 SQUARE by two L8† sets in which more than forty of the sixty-four pieces are embedded.



A tiling of the 24 x 24 SQUARE by two L8† sets. All of the pieces of one set and eight of the other are embedded.

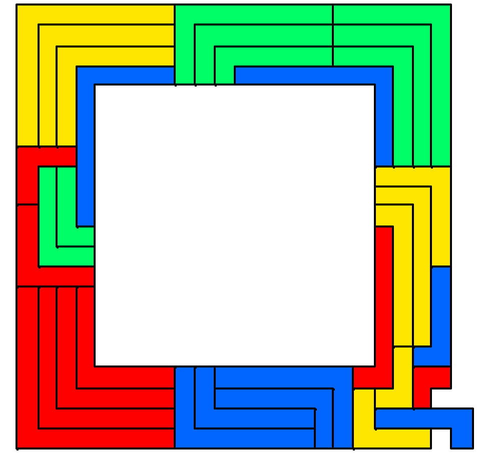
Values of the minimum number m of standard LOMINOES sets Ln whose combined area $mA(n)$ is equal to an integer squared are listed below for every set Ln in the interval $2 \le n \le 100$ for which $m \le 100$:

n	2	3	4	5	7	8	9	10	17	19	24	25	26	49	50	99
m	3	3	30	15	42	7	10	55	17	95	69	78	39	3	51	11

Klarner's theorem [HON 1976]:

"An a x b rectangle R can be packed with c x d rectangles if and only if either
(i) each of c,d divides one of a,b, each a different one, or
(ii) both c and d divide the same one of a,b, say a, and the other (b) is of the form b=cx+dy, for some non-negative integers x and y."

56. Square ANNULI tiled by one L8† set



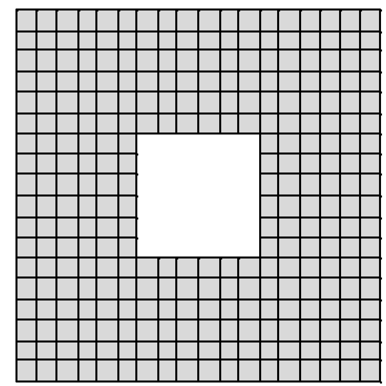
The 22^2-14^2 square annulus *not quite* tiled by one L8† set

Is there a tiling of the 22^2-14^2 square ANNULUS by one L8† set? A near miss is shown at the right. If you find a solution, please send a drawing of it to lominoes@neondsl.com. If you prove that the tiling is impossible, that information would be welcome!

18^2-6^2 (illustrated at lower right) is an arena for L8† that presents a much easier challenge than 22^2-14^2. One may argue that the reason for this is that it has a smaller 'I.Q.'. The I.Q. is the classical *isoperimetric quotient*:

$$\text{I.Q.} = (\text{total boundary length})/(\text{area})^{1/2} \qquad (56.1)$$

which is dimensionless and therefore independent of scale. It is probably true that plane tilings either by LOMINOES or by other varieties of polyominoes tend to have more solutions and are therefore easier to discover, the smaller the I.Q. of the tiling arena. (Can you explain why this is likely to be true?)



18^2-6^2 tiling arena

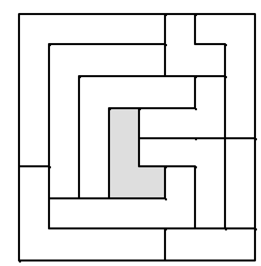
Exercises

1. Tile the 18^2-6^2 square ANNULUS with one L8† set.

2. Find a tiling of the 24 x 24 SQUARE (*cf.* p. 77) with two L8† sets in a pattern with C2 symmetry (*i.e.,* invariant under a half-turn about the center).

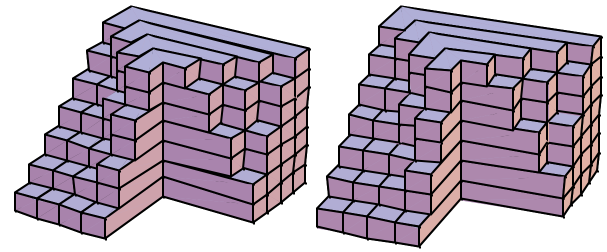
3. Demonstrate that the square hole in a tiling of the 8^2 - 2^2 square ANNULUS by the ten LOMINOES of L5 (*cf.* p. 70) can be moved to all possible positions in the tiling.

4. The square hole in tilings of the 8^2 - 2^2 square ANNULUS by the ten LOMINOES of L5 (*cf.* p. 70) can be replaced by a hole of the same area (four) in the shape of the outline of the piece [2,3]. Identify other examples of standard sets Ln that admit tilings of a SQUARE that contain one or more separated interior holes shaped like pieces of the set (*cf.* Eric Harshbarger's 'Enclosing Extra Pieces' pentomino designs [HAR 2005]).



57. Skip/Glide rule for partitioning L*n* into subsets whose volumes define an arithmetic sequence

Let $V = \lfloor n/2 \rfloor$. Skip/Glide is a rule that generates $N(n)/V$ subsets of L*n*, each of which contains V pieces. The subset volumes define an arithmetic sequence with increment V. We describe below the application of the rule to L8, where the sequence of seven subset volumes is {24, 28, 32, 36, 40, 44, 48).



The seven Skip/Glide subsets of L8 stacked in layers, with the smallest at the top and the largest at the bottom

(cross-eyed stereogram)

(1) First select the LOMINO [2,2] in the upper left corner of the Triangular Array (*cf.* p. 80). Next, advance (east) in the array in three steps of [2,0] each and combine the pieces at the four visited sites – [2,2], [2,4], [2,6], [2,8] – to form the smallest subset (top layer of the stack shown at the right).

(2) Select [2,3], at the upper-left-most position not yet visited, advance (east) in two steps of [2,0] each, followed by one of [1,-1] (southeast), and combine the pieces at the visited sites to form the second subset.

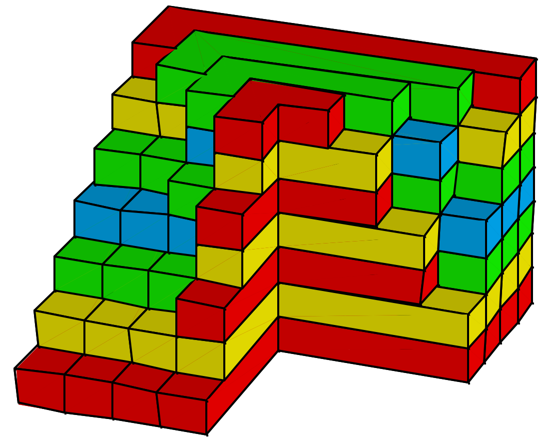
(3) Select [3,3], at the upper-left-most position not yet visited, advance (east) in two steps of [2,0] each, followed by one of [1,-1] (southeast), and combine the pieces at the visited sites to form the third subset.

(4) Select [3,4], at the upper-left-most position not yet visited, advance (east) in one step of [2,0], followed by two of [1,-1] (southeast), and combine the pieces at the visited sites to form the fourth subset.

Continue in this fashion until all seven subsets are formed. At each step, advance is to the right (east) unless the targeted site has already been visited – in which case advance is in the [1,-1] direction (southeast). 'Skip' describes a [2,0] step; 'glide' describes a [1,-1] step. The pieces in each of the seven stacked layers shown above are identified by color in the illustration on p. 80.

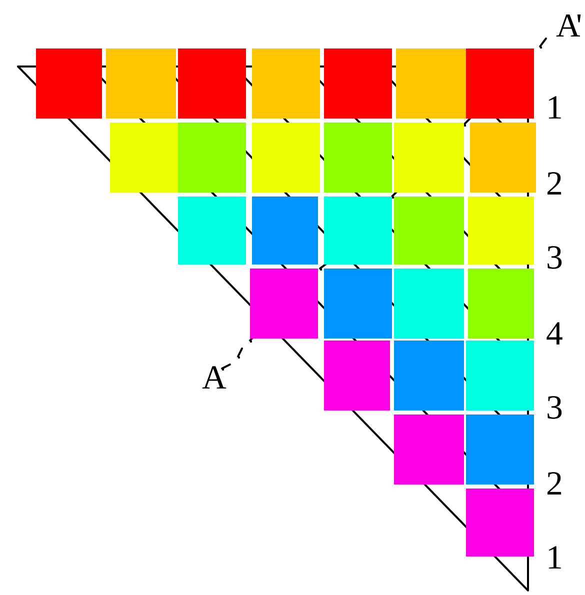
58. The seven Skip/Glide subsets of L8

In the assembly at the right, the seven four-piece subsets of L8 that are defined by Skip/Glide are shown stacked in layers in increasing order of volume from the top down. Although the volume v_k of the k^{th} of these seven subsets is equal to the volume of the corresponding 4-ring of the ZIGGURAT ${}_1[8|7,13]_1$, *it is impossible to arrange the pieces in every subset so as to form a set of tilings of seven 4-rings each of volume* v_k. If this were not the case, it would of course be trivially easy to construct a packing of the regular standard ZIGGURAT ${}_1[8|7,13]_1$. It is found that the only Skip/Glide subset that tiles the corresponding 4-ring for every n is the subset that is second largest in volume.



Skip/Glide stack for L8
The pieces are canonically colored

The colors in the array at the right define the composition of each of the seven four-piece Skip/Glide subsets for L8. The four LOMINOES in each subset are represented by squares of the same color. The numbers in the column at the right identify the pronic subsets that are composed of the pieces in the adjacent NW/SE strip.



The Skip/Glide construction is applicable to every standard LOMINOES set Ln.

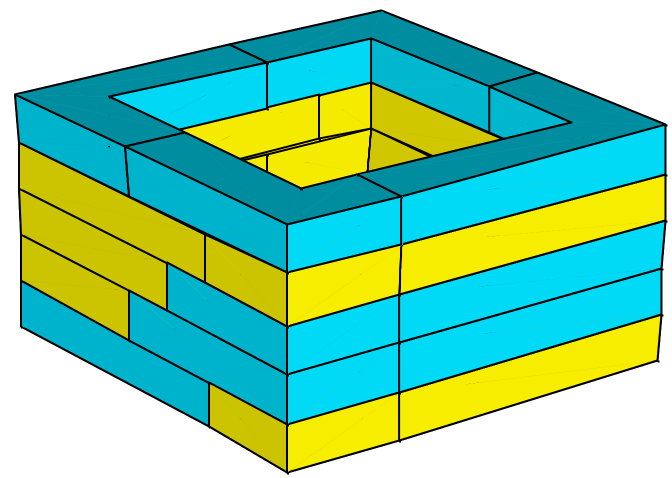
59. FAWLTY TOWERS

On pp. 17-19 of his remarkable book about polyominoes, "Puzzles, Patterns, Problems, and Packings", Solomon Golomb [GOL 1994] describes the history of the masonry-inspired problem of finding dominoes tilings that are free of 'fault lines'. Below is an example of the two-set TOWER $T_2[5|7,7]_5$ in which a single vertical *fault plane* extends from top to bottom through one wall of the TOWER. We call such a structure a FAWLTY TOWER

Can you explain why neither $T_1[8|10,10]_7$ nor $T_1[8^\dagger|10,10]_8$ can be assembled as a FAWLTY TOWER? Can you rearrange the pieces of the TOWER shown below so that the fault plane lies at some other distance from the corners of the TOWER? For what other values of n can you find examples of FAWLTY TOWERS? (Suggestion: first investigate the TOWER $T_2[4|6,6]_3$ composed of two L4 sets.)

The signatures of the 4-rings in the two-set L5 TOWER $T_2[5|7,7]_5$ illustrated here, reading from the top down, are:

<[4,2][5,3][4,3][4,3]>
<[2,2][5,2][5,2][5,5]>
<[3,2][5,4][3,3][4,4]>
<[4,2][5,3][4,4][3,3]>
<[2,2][5,5][2,3][4,5]>



A FAWLTY TOWER
$T_2[5|7,7]_5$

60. A conjectured infinite set ‘E’ of ZIGGURAT COMPLEXES for which r is even

We now describe **E** (which stands for **even**), a conjectured infinite set of *irregular* ZIGGURAT COMPLEXES for which r is even. It is conjectured that packings exist for every member of **E** except the smallest member, $_1[9|3,14]_1$. No packing can exist for $_1[9|3,14]_1$ because of the impossibility of simultaneously packing z-rings of ringwidths three and four (*cf.* pp. 47, 51). The total number of LOMINOES (595) in $_4[35|12,51]_{17}$, which is the second smallest member of **E**, is so forbiddingly large that no attempt has been made to prove the existence of a packing*. The justification for considering **E** is its relevance to the question of whether the number of z-rings in every irregular ZIGGURAT is necessarily odd, a possibility suggested by the three examples of irregular solitary ZIGGURATS ($_1[5|5,7]_1$, $_1[8|4,12]_1$, and $_1[11|4,19]_1$) for which packings are known (*cf.* pp. 30 and 34).

Recall that $_1[8|4,12]_1$ is the smallest member of an infinite family of irregular ZIGGURAT COMPLEXES $_1[8k|4k,12k]_k$ derived from the regular medial ZIGGURAT COMPLEXES $_1[8k|4k+3,12k+1]_k$ (*cf.* p. 54). It is unknown whether the irregular ZIGGURATS $_1[5|5,7]_1$ and $_1[11|4,19]_1$ also belong to infinite families whose values of n, a, and b are defined by a linear (or other polynomial) function of an integer parameter. The ZIGGURAT COMPLEXES in **E**, which are described on pp. 83-85, are defined by values of n that are defined by a *quadratic* function of an integer parameter k.

_____

*The search for packings of ZIGGURATS belongs to the class of problems called NP-complete. The known algorithms for solving problems in this complexity class depend exponentially on the size of the problem. The only known method of finding every solution is to test every possible combination of pieces, but this is extremely time-consuming for problems of large size.

For every ZIGGURAT COMPLEX ${}_q[n\,|\,a,b]_s$ in **E**,

$$n=8k^2+2k-1$$
$$=(4k-1)(2k+1) \qquad (k=1, 2,\ldots); \qquad (60.1)$$

$$a=4k^2+k+1 \qquad (k=1, 2,\ldots); \qquad (60.2)$$
$$b=12k^2+3k-1 \qquad (k=1, 2,\ldots). \qquad (60.3)$$

The *progenitor* of ${}_q[n\,|\,a,b]_s$ is the regular standard ZIGGURAT COMPLEX ${}_q[n\,|\,a',b']_s$, where

$$a'=4k^2-2k \qquad (k=1, 2,\ldots); \qquad (60.4)$$
$$b'=12k^2+2k-1 \qquad (k=1, 2,\ldots). \qquad (60.5)$$

The number $\boldsymbol{r}'$ of 4-rings in ${}_q[n\,|\,a',b']_s$ is $\boldsymbol{r}'=n=(4k-1)(2k+1)$ $(k=1, 2,\ldots)$; (60.6)
the number $\boldsymbol{r}$ of z-rings in ${}_q[n\,|\,a,b]_s$ is $\boldsymbol{r}=n+2k+1$
$=4k(2k+1)$ $(k=1, 2,\ldots)$. (60.7)

For both ${}_q[n\,|\,a',b']_s$, and ${}_q[n\,|\,a,b]_s$, q=the number of Ln sets and s=the number of ZIGGURATS in the ZIGGURAT COMPLEX.

For $n\equiv 1 \pmod 8$, *i.e.*, $k\equiv 1 \pmod 4$: $q=1$ and $s=1+(n-9)/8$; (60.8)
For $n\equiv 3 \pmod 8$, *i.e.*, $k\equiv 2 \pmod 4$: $q=4$ and $s=1+(n-3)/2$; (60.9)
For $n\equiv 5 \pmod 8$, *i.e.*, $k\equiv 3 \pmod 4$: $q=2$ and $s=1+(n-5)/4$; (60.10)
For $n\equiv 7 \pmod 8$, *i.e.*, $k\equiv 0 \pmod 4$: $q=4$ and $s=1+(n-7)/2$. (60.11)

The values of s may also be expressed as follows:

For $k \equiv 1 \pmod 4$: $s=16k^2-23k+8$

$$=16\left[k-(23-\sqrt{17})/32\right]\left[k-(23+\sqrt{17})/32\right]. \qquad (60.12)$$

For $k \equiv 2 \pmod 4$: $s=64k^2-60k+13$

$$=64\left[k-(15-\sqrt{17})/32\right]\left[k-(15+\sqrt{17})/32\right]. \qquad (60.13)$$

For $k \equiv 3 \pmod 4$: $s=32k^2-14k+1$

$$=32\left[k-(7-\sqrt{17})/32\right]\left[k-(7+\sqrt{17})/32\right]. \qquad (60.14)$$

For $k \equiv 0 \pmod 4$: $s=64k^2+4k-1$

$$=64\left[k-(-1-\sqrt{17})/32\right]\left[k-(-1+\sqrt{17})/32\right]. \qquad (60.15)$$

The number $N(n)$ of LOMINOES per set, the set volume $V_{\text{set}}(n)$, and the volume of a set of z-rings of consecutive ringwidths $\boldsymbol{a}+1, \boldsymbol{a}+2, \ldots, \boldsymbol{b}, \boldsymbol{b}+1$ may be expressed as polynomial functions of k:

$$N(n)=n(n-1)/2$$
$$=(4k-1)(2k+1)(4k^2+k-1) \qquad (k=1, 2,\ldots); \qquad (60.16)$$

$$V_{\text{set}}(n)=n(n^2-1)/2$$
$$=2k(4k-1)(4k+1)(2k+1)(4k^2+k-1) \qquad (k=1, 2,\ldots); \qquad (60.17)$$

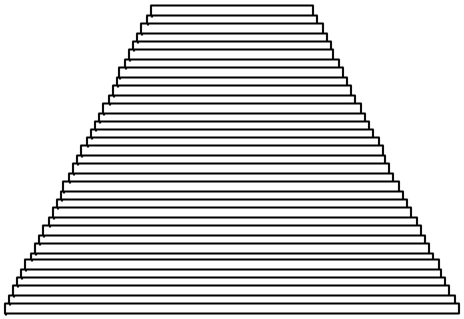
$$V_{\text{rings}}(\boldsymbol{a}, \boldsymbol{b})=2(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{b}-\boldsymbol{a}+1)$$
$$=8k(4k-1)(4k+1)(2k+1) \qquad (k=1, 2,\ldots). \qquad (60.18)$$

k	n	n (mod 8)	$[a, b]$	$[a', b']$	q	s	r	r'	$\langle z \rangle_{Av}$	$V_{rings}(a, b)$	$V_{set}(n)$
1	9	1	[2,13]	[6,14]	1	1	12	9	3	360	360
2	25	3	[12,51]	[19,53]	4	17	40	35	3.5	5040	21420
3	77	5	[30,113]	[40,116]	2	19	84	77	~3.667	24024	228228
4	135	7	[56,199]	[69,203]	4	67	144	135	3.75	73440	1230120
5	209	1	[90,309]	[106,314]	1	26	220	209	3.8	175560	4564560
6	299	3	[132,443]	[151,449]	4	149	312	299	~3.833	358800	53461200
7	405	5	[182,601]	[204,608]	2	101	420	405	~3.857	657720	66429720
8	527	7	[240,783]	[265,791]	4	263	544	527	3.875	1113024	292725312

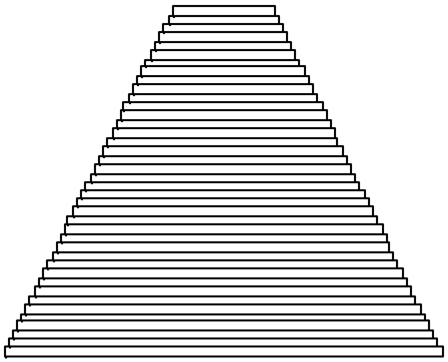
The above table contains data for the first eight members of **E**. $\langle z \rangle_{Av}$ is the average number of LOMINOES per z-ring for each ZIGGURAT in the ZIGGURAT COMPLEX.

> The volume of the k largest 4-rings of every progenitor ZIGGURAT COMPLEX ${}_q[n\,|\,a',b']_s$ is equal to the volume of the $3k$+1 smallest z-rings of the E ZIGGURAT COMPLEX ${}_q[n\,|\,a,b]_s$.

The second member of **E**, ${}_4[35\,|\,12,51]_{17}$, and its progenitor, ${}_4[35\,|\,19,53]_{17}$, are shown at the right.



${}_4[35\,|\,19,53]_{17}$
r' =35

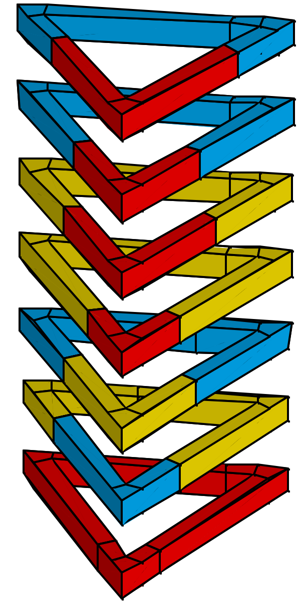


${}_4[35\,|\,12,51]_{17}$
r =40

61. $_p$LOMINOES: LOMINOES with turning angle $2p/p$ (p=3,4,…)

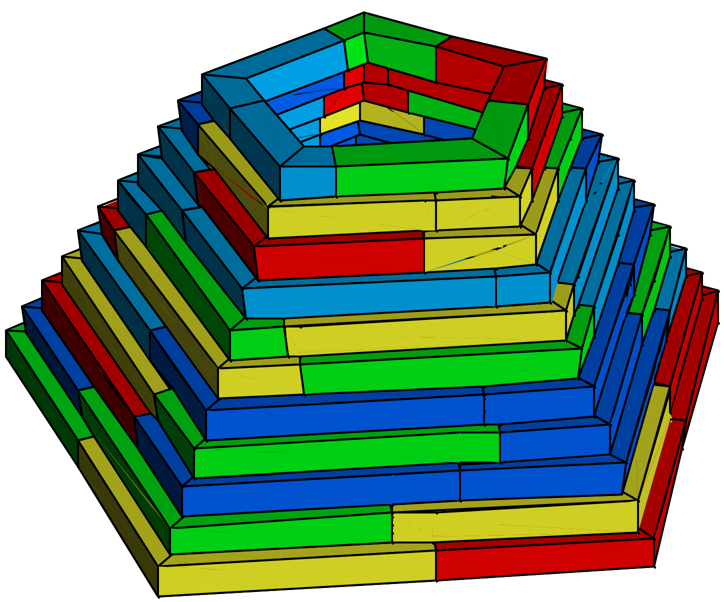
The angle between the two arms of an ordinary right-angled LOMINO is 90°. But we can expand the definition of LOMINO to include sets of pieces with inter-arm angles different from 90°. We define a $_p$LOMINO as a LOMINO analog for which the turning angle is $2p/p$ and the inter-arm angle is consequently $(1-2/p)p$ (p=3,4,…). $_p$LOMINOES for p=4 are ordinary right-angled LOMINOES, and we will continue to call them simply LOMINOES. Analogs of SAWTOOTHS, FENCES, TOWERS, ZIGGURATS, and SKYSCRAPERS tiled by $_p$LOMINOES exist for $p \neq 4$. Tilings and packings for examples of these structures for several values of n have been found for p=3, 5, and 6. Examples for p=3 and p=5 are shown here.

The TOWER $T_1[_37|9,9]_7$ (right) is composed of the set $_3$L7 of $_3$LOMINOES; its seven triangular 3-rings are of ringwidth nine. The eleven pentagonal rings of the $_5$ZIGGURAT $_1[_511|8,18]_1$, composed of one $_5$L11 set (p=5), have ringwidths eight to eighteen. p-gonal ring signatures for the packings of these two structures are listed below.



$T_1[_37|9,9]_7$
One of the four solutions, canonically colored

[7,4][5,5][4,2]
[7,5][4,4][5,2]
[7,3][6,6][3,2]
[7,6][3,3][6,2]
[6,4][5,4][5,3]
[4,3][6,3][6,5]
[7,7][2,2][7,2]

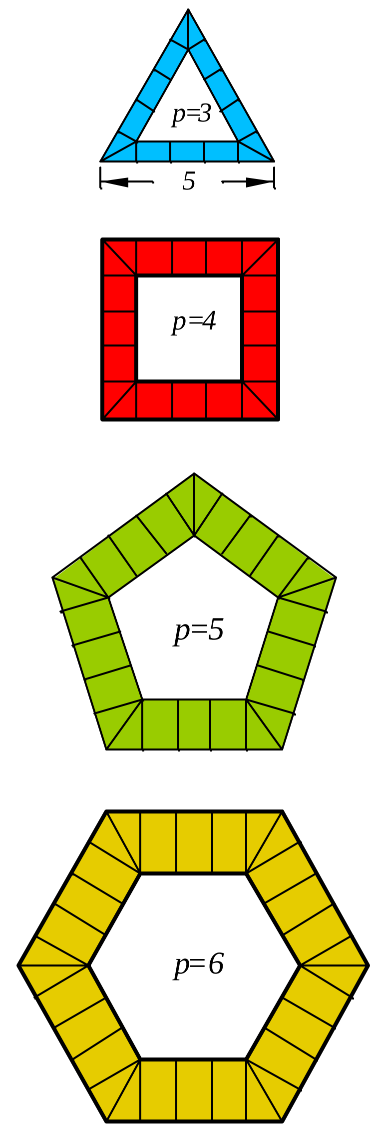


$_1[_511|8,18]_1$

<[2,4][4,4][4,6][2,5][3,6]>
<[2,2][7,7][2,3][6,7][2,7]>
<[3,3][7,5][5,4][6,6][4,7]>
<[3,4][7,3][8,2][9,6][5,8]>
<[3,8][4,10][2,10][2,9][3,9]>
<[2,6][7,10][3,10][3,11][2,11]>
<[3,5][9,5][9,4][10,6][8,11]>
<[5,6][9,7][8,4][11,4][11,10]>
<[5,5][11,5][11,6][10,5][11,11]>
<[6,8][9,9][8,9][8,10][7,11]>
<[7,8][10,10][8,8][10,9][9,11]>

62. The height h_p and thickness t_p of a ${}_p$LOMINO

Examples of p-gonal rings of ringwidth five are shown in plan view at the right. We denote by r_1 and r_2 the inradii of the inner and outer regular p-gon boundaries of the ring, respectively. The *thickness* of each ${}_p$LOMINO is defined as $t_p=r_2-r_1$. It follows from the geometry of the triangular prisms in the ring corners that $r_1=t_p\,(w-2)/2$ and $r_2=t_p\,w/2$, where $t_p=\cot(\mathbf{p}\;/p)$ and w is the ringwidth. For esthetic reasons the transverse cross-section of the arms of each ring is chosen to be square. Hence the *height* h_p of each ${}_p$LOMINO, which is measured parallel to the axis of p-fold symmetry of the p-gonal ring, is equal to its thickness t_p.

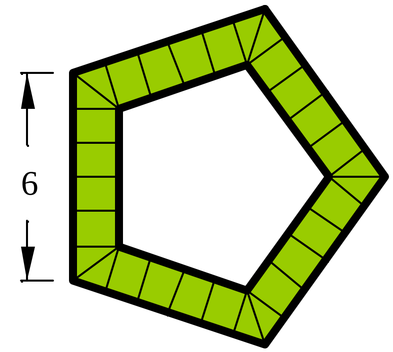


p-gonal p-rings
of ringwidth 5

p	$t_p\,(=h_p)$		$r_1/(w\text{-}2)$	r_2/w
3	$\sqrt{3}/3$	$\cong$ 0.5774	~0.2887	~0.2887
4		1	0.5	0.5
5	$\sqrt{2}\left(5+\sqrt{5}\right)^{3/2}/20$	$\cong$1.376	~0.6882	~0.6882
6	$\sqrt{3}$	$\cong$ 1.732	~0.8660	~0.8660
7		~ 2.076	~1.038	~1.038
8	$1+\sqrt{2}$	$\cong$ 2.414	~1.207	~1.207
9		~ 2.748	~1.374	~1.374

63. Solitary regular standard $_p$ZIGGURAT $_1[_pn\,|\,a,b]_1$ (p = 3, 4, 5, …)

A solitary regular standard $_p$ZIGGURAT is a single collimated stack of r p-gonal rings of consecutive ringwidths from a to b inclusive, composed of the $n(n\text{-}1)/2$ $_p$LOMINOES of $_p$Ln; $p=\lfloor n/2 \rfloor$. The values of a and b may be derived by comparing a $_p$ZIGGURAT and a $_p$TOWER that have the same average ringwidth $w_{Av}=n+2$ (*cf.* p. 56).



pentagonal ring
of ringwidth 6

Let us consider first the case where n is even and $p=n/2$. The total number of $_p$LOMINOES in $_p$Ln is equal to the product of the number r of p-gonal rings in the $_p$ZIGGURAT or $_p$TOWER and the number p of $_p$LOMINOES in each ring:

$$n(n\text{-}1)/2 = r\ p$$

$$= r\ n/2. \qquad (63.1)$$

Hence

$$r = n\text{ -}1. \qquad (63.2)$$

There are $(r\text{ -}1)/2$ p-gonal rings above the central p-gonal ring of the $_p$ZIGGURAT or $_p$TOWER and $(r\text{ -}1)/2$ p-gonal rings below it. Since the ringwidths of consecutive rings of the $_p$ZIGGURAT differ by one, the ringwidths a and b of the top and bottom rings are related to the ringwidth w_{Av} $(=n+2)$ of the central ring as follows:

$$a = w_{Av} \text{ - } \Delta w \qquad (63.3)$$

$$b = w_{Av} + \Delta w, \qquad (63.4)$$

where

$$\Delta w = (r\text{ -}1)/2. \qquad (63.5)$$

Substituting for r from Eq. 63.2 yields

$$\Delta w = (n\text{ -}2)/2. \qquad (63.6)$$

Hence

$$a = (n+6)/2 \qquad (63.7)$$

For odd n, an analysis similar to the one above yields

$$r = n \tag{63.9}$$
$$a = (n+5)/2 \tag{63.10}$$
$$b = (3n+3)/2. \qquad \text{(odd } n\text{)} \tag{63.11}$$

The general case, which includes p=4, can therefore be described by the equations

$$\boxed{\begin{aligned} a &= \lfloor (n+6)/2 \rfloor \\ b &= \lfloor (3n+3)/2 \rfloor . \end{aligned}} \qquad \text{(odd or even } n\text{)} \qquad \begin{aligned} &(63.12) \\ &(63.13) \end{aligned}$$

Whether a packing exists for a particular $_p$ZIGGURAT can be determined only by a detailed investigation. It has been proved by a full tree search, for example, that no packing exists for the five-story $_3$ZIGGURAT $_1[_3 6 \,|\, 6{,}10]_1$.

$_p$ZIGGURATS for $6 \le n \le 15$ are shown on p. 98-100. Packings have been found for the following cases:

$$_1[_3 7 \,|\, 6{,}12]_1$$
$$_1[_4 8 \,|\, 7{,}13]_1$$
$$_1[_4 9 \,|\, 7{,}15]_1$$
$$_1[_5 11 \,|\, 8{,}18]_1$$

It is conjectured that packings exist for every solitary regular standard $_p$ZIGGURAT for n>15.

The cross-section area $\kappa_4 = h_4 t_4$ of every ${}_4$LOMINO in L8 or L8† is equal to one, since both the height h_4 and thickness t_4 of every ${}_4$LOMINO are equal to one (*cf.* p. 87). For $p \neq 4$, however, the cross-section area $\kappa_p = \cot^2(\pi/p) \neq 1$. Consequently, for $p \neq 4$ Eqs. 15.3 and 15.5 (*cf.* p. 31) must be replaced by Eqs. 63.14 and 63.15:

$$ {}_pV_{\text{set}}(n) = \kappa_p\, n(n^2-1)/2 = \text{volume of } {}_p\text{L}n. \tag{63.14}$$

$$\begin{gathered}\text{Volume } {}_pV_{\text{rings}}(a,b) \text{ of } r \text{ } p\text{–gonal rings of ringwidths } a, a+1, \ldots, b-1, b \\ = 2\kappa_p(\beta - \alpha + 1)(\alpha + \beta), \\ \text{where } r = \beta - \alpha + 1,\ \alpha = a-1, \text{ and } \beta = b-1. \end{gathered} \tag{63.15}$$

Furthermore, the following expressions for the ${}_p$TOWER parameters q and s, which replace Eqs. 17.1 and 17.2 (*cf.* p. 33), apply to all $p \geq 3$:

$$q = lcm[\mathrm{N}(n), p]/\mathrm{N}(n) \tag{63.16}$$

$$s = q\mathrm{N}(n)/p \tag{63.17}$$

The parameters q and s for a regular ${}_p$ZIGGURAT COMPLEX ${}_q[{}_pn \mid a,b]_s$ for $p \neq 4$ are (*cf.* p. 32):

$$q = lcm[{}_pV_{\text{set}}(n),\ {}_pV_{\text{rings}}(a,b)]/\ {}_pV_{\text{set}}(n) \tag{63.18}$$
$$s = lcm[{}_pV_{\text{set}}(n),\ {}_pV_{\text{rings}}(a,b)]/\ {}_pV_{\text{rings}}(n) \tag{63.19}$$

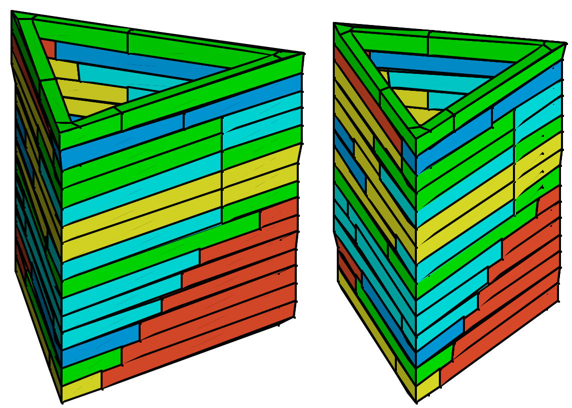
For solitary ${}_p$TOWERS and ${}_p$ZIGGURATS (*cf.* examples on p. 86), $n \equiv 0$ or 1 (mod $2p$).

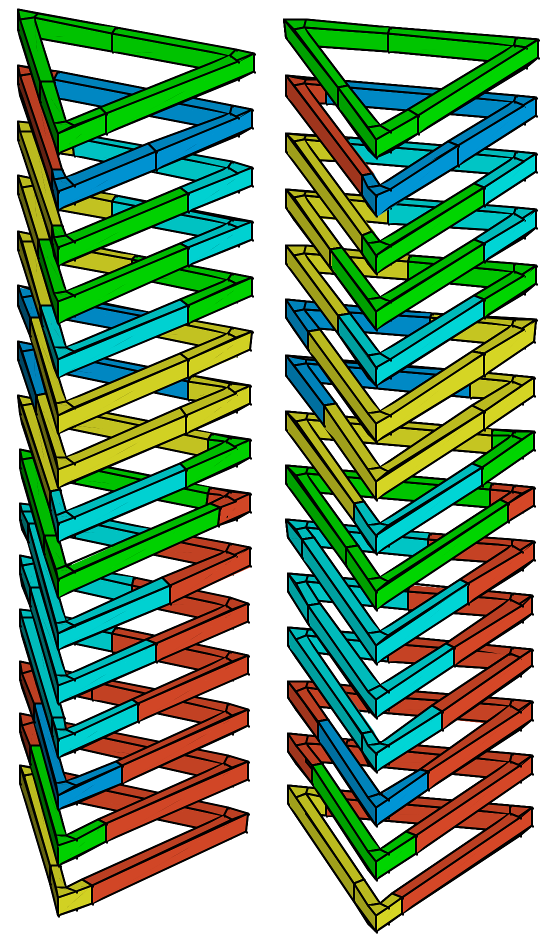
There is no special advantage in defining augmented sets for $p \neq 4$. Moreover, *irregular* ZIGGURATS are undefined for $p \neq 4$, since every tiling of a polygonal ring for $p \neq 4$ requires p ${}_p$LOMINOES.

64. A trigonal TOWER for n=10

It is not essential that $p=\lfloor n/2 \rfloor$ for a solitary regular standard $_p$ZIGGURAT or $_p$TOWER (*cf.* p. 88). Consider the fifteen-story canonically colored $_3$TOWER $T_1[_310|12,12]_{15}$, composed of the 45 pieces of the set $_3$L10, which is illustrated here in two stereograms. The signatures of this $_3$TOWER's fifteen triangular rings are listed at left.

<[5,3][9,7][5,7]>
<[2,6][6,10][2,10]>
<[2,8][4,9][3,10]>
<[6,8][4,7][5,6]>
<[5,8][4,6][6,7]>
<[9,8][4,5][7,3]>
<[7,8][4,3][9,5]>
<[3,8][4,2][10,9]>
<[4,10][2,2][10,8]>
<[10,7][5,5][7,2]>
<[9,6][6,6][6,3]>
<[2,5][7,7][5,10]>
<[8,4][8,8][4,4]>
<[9,3][9,9][3,3]>
<[3,2][10,10][2,9]>



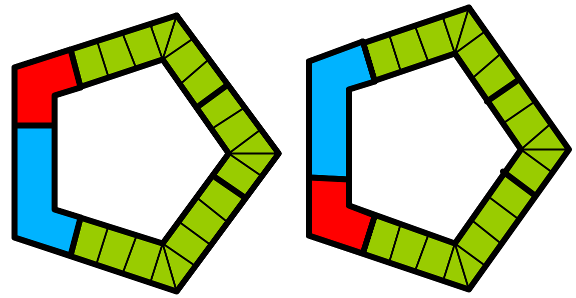


It is not difficult to prove that no packing exists for the corresponding fifteen-story solitary $_3$ZIGGURAT $_1[_310|5,19]_1$. The only possible tiling of the smallest triangular ring (ringwidth five) is <[2,2][3,2][3,3]>. But every one of the three possible tilings (below) of a ring with the next larger ringwidth, six, requires at least one of the pieces in this smallest ring:

<[2,2][4,2][4,4]>, <[2,3]3,3][3,4]>, <[2,4][2,3][3,4]>. □

65. Multiplicity polynomials for $p = 3$, 4, and 5

Examples of degenerate 4-rings were described on p. 4. Similar degeneracies occur for $p>4$. An example of a doubly degenerate ring for $p=5$ is shown at the right. By contrast, every triangular ring for $p=3$ admits only one arrangement of the three $_3$LOMINOES that tile it. As a consequence, it is not difficult to derive exact expressions for the multiplicities $_3M_{odd}[w]$ and $_3M_{even}[w]$, especially if one uses the Polya-Burnside formula. Golomb [GOL 1999] has described a variety of problems in enumerative combinatorics that are readily solved by means of this formula. Here is Golomb's description of Polya-Burnside:



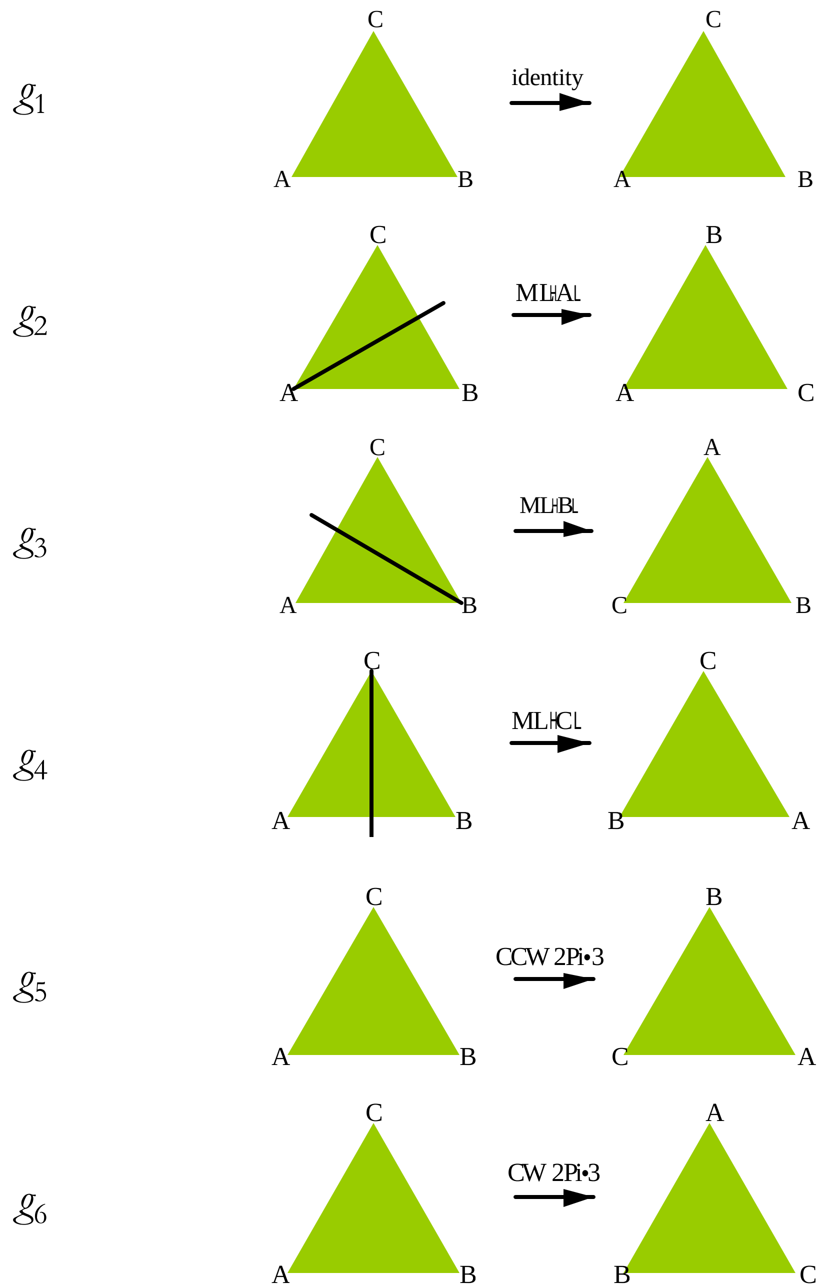
An example of a pentagonal ring for $p=5$ that admits two different *arrangements* for the same *composition*.

> 'Let S be any finite collection of objects, and let G be a finite group of symmetries for these objects, with n symmetry operations g_1, g_2, …, g_n comprising G. (One of these symmetry operations must be the identity operator.) Let $C(g)$ describe the number of objects in the collection, S, left fixed by the symmetry, g, of G. Then the number, N, of objects in S *distinguishable relative to the symmetries of G* is given by
>
> $$N = \frac{1}{n}[C(g_1) + C(g_2) + \ldots + C(g_n)]\text{'} \qquad (65.1)$$

For triangular rings tiled by $_3$LOMINOES, G is the group D3 comprised of the six symmetries of an equilateral triangle. We label the three vertices of the triangle A, B, and C.

g_1 = identity operator;
g_2 = *reflection* in mirror line ML(A) incident at vertex A;
g_3 = *reflection* in mirror line ML(B) incident at vertex B;
g_4 = *reflection* in mirror line ML(C) incident at vertex C;
g_5 = CCW *rotation* of $2\pi/3$ about the triangle center;
g_6 = CW *rotation* of $2\pi/3$ about the triangle center;

These six symmetries are illustrated on p. 93. We prove on pp. 94-96 that $C(g_2)=C(g_3)=C(g_4)=C(g_5)=C(g_6)=0$.



The six symmetry operations of an equilateral triangle

Proof that that $C(g_2)=C(g_3)=C(g_4)=C(g_5)=C(g_6)=0$

For a tiling of a triangular ring of even ringwidth to be left fixed by one of the three reflections g_2, g_3, or g_4, the two 3LOMINOES in the tiling that are separated by the line of reflection would have to be identical. Since there are no identical pieces in the set 3Ln, such an arrangement is impossible ('disallowed'). It is also impossible for a tiling of a triangular ring of odd ringwidth to be invariant under reflection. (Why?) Hence for both even and odd w,

$$C(g_2)=C(g_3)=C(g_4)=0 \qquad (65.2)$$

If a tiling of a triangular ring of either even or odd ringwidth were invariant under rotational symmetry operation g_5 and g_6, all three 3LOMINOES in the tiling would have to be identical. Hence for both even and odd w,

$$C(g_5)=C(g_6)=0. \qquad (65.3)$$

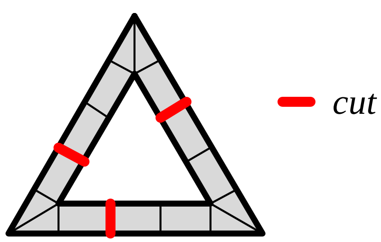
Substituting Eqs. 65.2 and 65.3 in Eq. 65.1, we conclude that

$$N=\frac{1}{6}[C(g_1)]. \qquad (65.4)$$

Now let

$$C(g_1)= C_{\text{unrestricted}} - C_{\text{disallowed}}\,. \qquad (65.5)$$

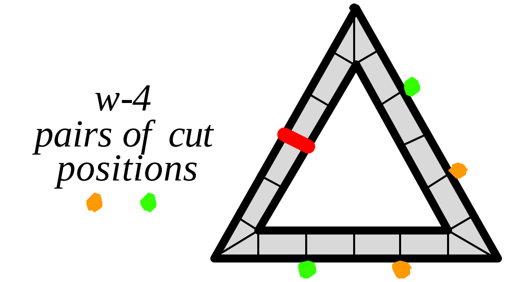
$C_{\text{unrestricted}}$ is the total number of ways three transverse *cuts* (*cf.* illustration at right) can be assigned to the three 'legs' of a triangular ring, without regard to whether any of the three pieces defined by the cuts are duplicated. The cut on each leg lies at integer distance from the ends of the leg. $C_{\text{disallowed}}$ is the number of *disallowed* arrangements, *i.e.,* those cut distributions in which either two or three of the pieces are identical.



We first count $C_{\text{unrestricted}}$.

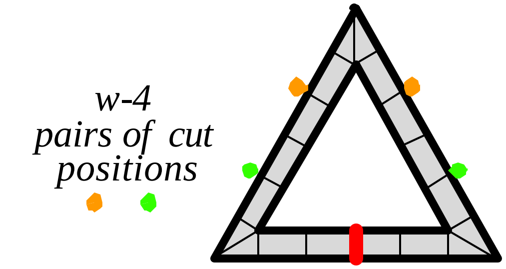
There are w-3 possible positions for a cut on one leg of a triangular ring of ringwidth w. Since the positions of the cuts on the three legs are independent, we conclude that for both even and odd w,

$$C_{\text{unrestricted}}=(w\text{-}3)^3 \qquad (65.6)$$



Next we count $C_{\text{disallowed}}$.

(a) **even** w

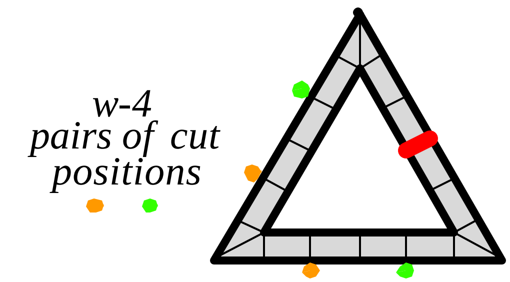


The disallowed arrangements are of two types – those in which

(i) two (but *not three*) pieces are identical,

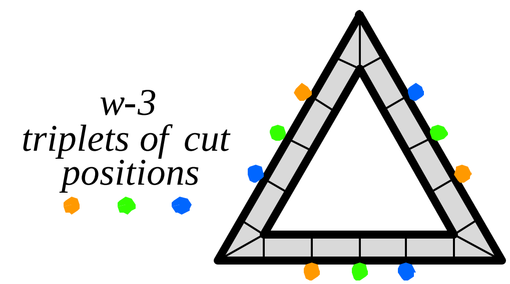
and those in which

(ii) all three pieces are identical.



(i) The cut on one of the three legs lies at the center of the leg. For the cuts on the other two legs, the leg center position is excluded. Hence there are w-4 possible pairs of positions for these other two cuts. Since the centrally positioned cut can be assigned to any one of the three legs, the total number of disallowed arrangements is equal to 3(w-4). These three sets of cases are illustrated at top right.

(ii) The disallowed arrangements are those in which the positions of the cuts on all three legs of the triangular ring are invariant under rotation g_5 and g_6. There are w-3 such disallowed cut distributions. They are illustrated at bottom right.



Summing the contributions to $C_{\text{disallowed}}$ from (i) and (ii), we find that for even w,

$$\begin{aligned} C_{\text{disallowed}} &= 3(w-4)+w-3 \\ &= 4w-15. \end{aligned} \tag{65.7}$$

If we substitute Eqs. 65.5-65.7 in Eq. 65.4, we conclude that for even w,

$$\begin{aligned} {}_3M_{even}[w] &= \frac{1}{6}(C_{\text{unrestricted}} - C_{\text{disallowed}}) \\ &= \frac{1}{6}[(w-3)^3 - (4w-15)] \\ &= \frac{1}{6}(w-2)(w-3)(w-4) - \frac{1}{2}(w-4). \end{aligned} \tag{65.8}$$

(b) **odd** w

There are no arrangements with reflection symmetry. The disallowed arrangements are those with three identical pieces, *i.e.,* tilings that are invariant under rotation g_5 and g_6. Using arguments similar to those employed above for even w, we find that

$${}_3M_{odd}[w] = \frac{1}{6}(w-2)(w-3)(w-4). \tag{65.9}$$

Eqs. 65.8 and 65.9 are in exact agreement with results computed for $5 \leq w \leq 20$, using an algorithm analogous to the algorithm for 4-rings described on p. 61.

Summary: for both p=3 and p=4, the values of ${}_pM_{odd}[w]$ and ${}_pM_{even}[w]$ are defined by the following polynomials of degree p:

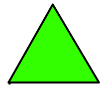
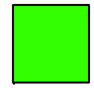
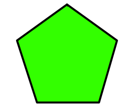
p=3 $\quad {}_3\mathrm{M}_{odd}[w]=(w-2)(w-3)(w-4)/6$ (65.10)

$\qquad\quad {}_3M_{even}[w]=(w-2)(w-3)(w-4)/6-(w-4)/2$

p=4 $\quad {}_4M_{odd}[w]=[(w-2)(w-3)(w-4)(w-5)+(w-3)(w-5)(w-7)]/8$ (65.11)

$\qquad\quad {}_4M_{even}[w]=[(w-2)(w-3)(w-4)(w-5)+(w-4)(w^2-13w+38)]/8$

The table below lists numerical values of ${}_pM_{odd}[w]$ and ${}_pM_{even}[w]$ for p=3, 4, and 5.

w	${}_3M[w]$	${}_3Z[w]$	$\Delta_3[w]$	${}_4M[w]$	${}_4Z[w]$	$\Delta_4[w]$	${}_5M[w]$	${}_5Z[w]$	$\Delta_5[w]$
5	1	1	0	0	0	--	0	0	--
6	3	4	.333	2	3	-.500	2	0	-.333
7	10	10	0	15	15	0	18	12	-.333
8	18	20	.111	44	45	-.023	97	72	-.258
9	35	35	0	111	105	.054	324	252	-.222
10	53	56	.057	216	210	.028	831	672	-.191
11	84	84	0	402	378	.060	1860	1512	-.187
12	116	120	.034	656	630	.040	3614	3024	-.163
13	165	165	0	1050	990	.057	6624	5544	-.163
14	215	220	.023	1550	1485	.042	10630	9504	-.106
15	286	286	0	2265	2145	.053	16328	15444	-.054
16	358	364	.017	3132	3003	.041	23058	24024	.042
17	455	455	0	4305	4095	.049	35163	36036	.025
18	553	560	.013	5684	5460	.039	51551	52416	.017
19	680	680	0	7476	7140	.045	--	74256	--
20	808	816	.010	9536	9180	.037	--	102816	--

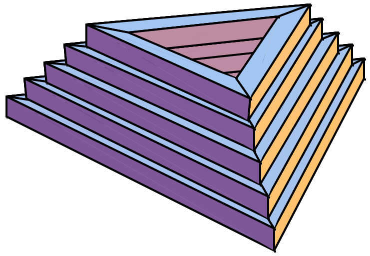
In contrast to the case of p=3 and p=4, the values of ${}_5M_{odd}[w]$ and ${}_5M_{even}[w]$ for p=5 in the interval $4 \leq w \leq 18$ cannot be expressed by a polynomial of degree p. Let us define

$$ {}_pZ[w] = (w\text{-}2)(w\text{-}3)\ldots(w\text{-}(p+1))/2p \qquad (65.12)$$

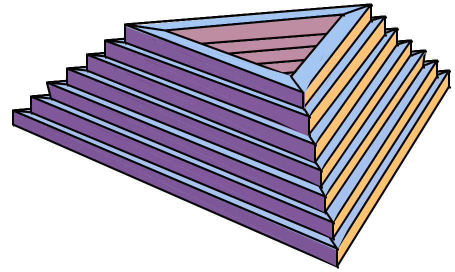
and

$$\Delta_p[w] = \frac{{}_pZ[w] - {}_pM[w]}{{}_pM[w]}. \qquad (65.13)$$

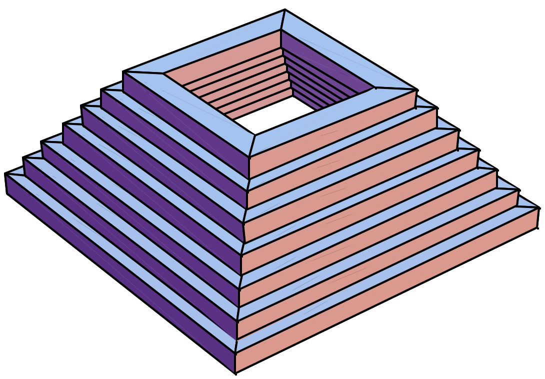
The tabulated data for $\Delta_3[w]$, $\Delta_4[w]$, and $\Delta_5[w]$ suggest the possibility that for all $p \geq 3$, $\lim_{w\to\infty} \Delta_p[w]=0$.



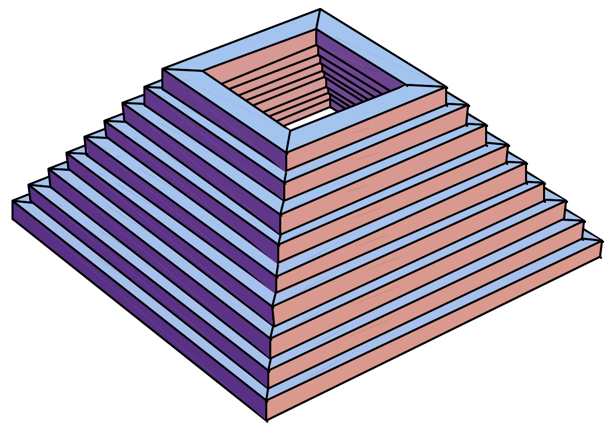
$_1[_3 6 \,|\, 6,10]_1$



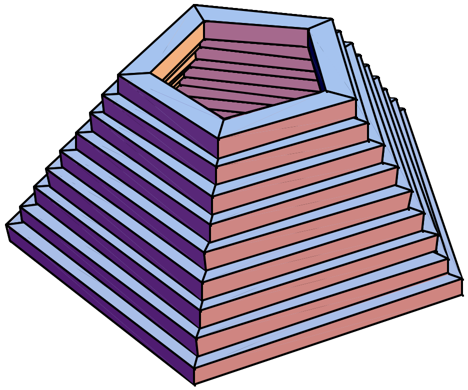
$_1[_3 7 \,|\, 6,12]_1$



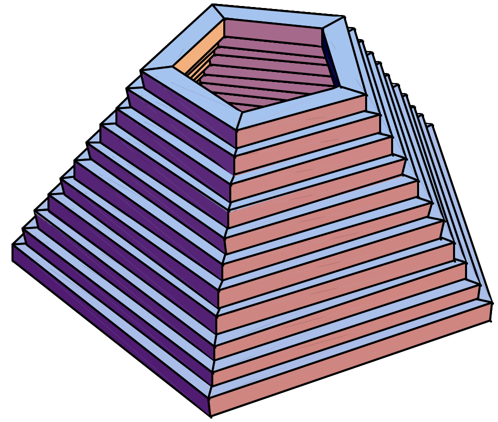
$_1[_4 8 \,|\, 7,13]_1$



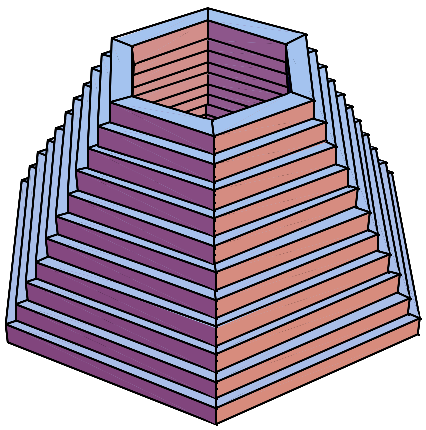
$_1[_4 9 \,|\, 7,15]_1$



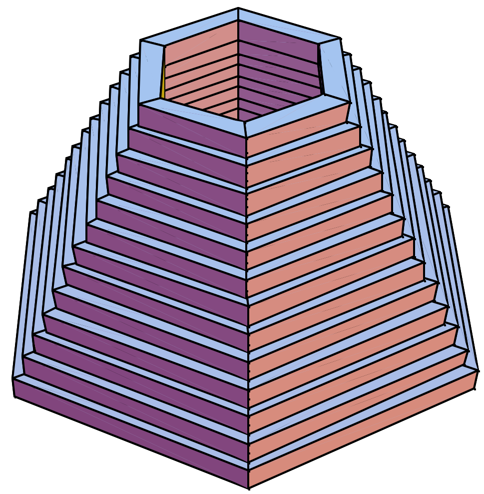
$_1[_5 10 \mid 8{,}16]_1$



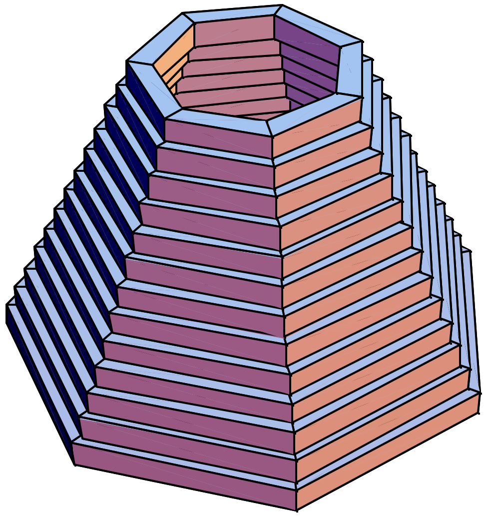
$_1[_5 11 \mid 8{,}18]_1$



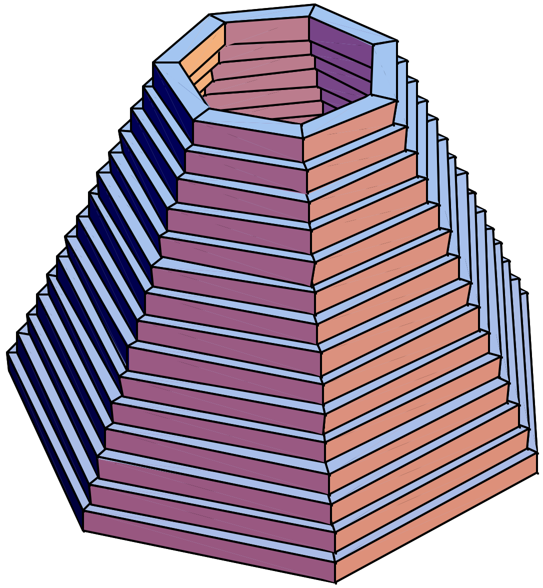
$_1[_6 12 \mid 9{,}19]_1$



$_1[_6 13 \mid 9{,}21]_1$

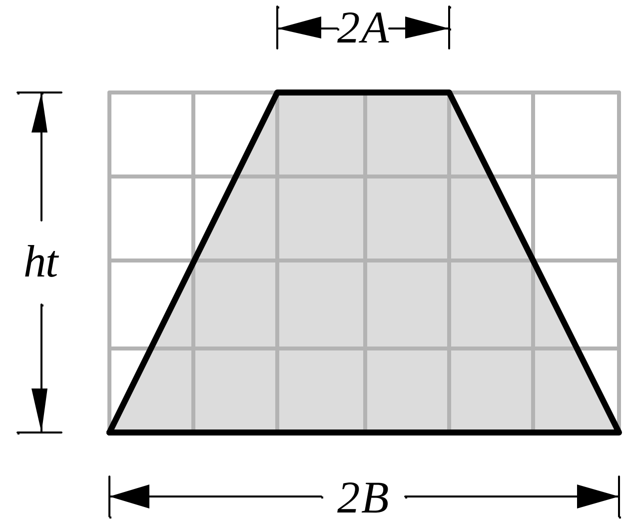


$_1[_7 14 \mid 10,22]_1$



$_1[_7 15 \mid 10,24]_1$

66. The asymptotic shape of a ${}_p$ZIGGURAT for which $p=\lfloor n/2 \rfloor$



Silhouette of asymptotic p-ZIGGURAT

What would be the shape of the silhouette of a solitary regular standard ${}_p$ZIGGURAT for which $p=\lfloor n/2 \rfloor$ if p – and therefore n – increased without limit? We denote the overall height of such an asymptotic ${}_p$ZIGGURAT by ht, and the inradii of the outer boundary of the top and bottom p-gonal rings by A and B, respectively. Let us determine the relative magnitudes $A : B : ht$.

We consider the case of even n. (In the limit, it doesn't make any difference whether n is odd or even.)

Since $p=n/2$ and $\rho = n - 1$, the height h_p of each p-gonal ring (*cf.* p. 88) is

$$h_p = \cot(2\pi / n) \qquad (66.1)$$

Therefore

$$ht = \rho\, h_p$$
$$= (n-1) \cot(2\pi / n) \qquad (66.2)$$

According to Eqs. 63.7 and 63.8, the top and bottom ringwidths of a ${}_p$ZIGGURAT for even n are

$$a = (n+6)/2 \qquad (63.7)$$
$$b = (3n+2)/2. \qquad (63.8)$$

Since the inradius r of the outer boundary of a p-gonal ring of ringwidth w (*cf.* p. 87) is

$$r = w \cot(2\pi / n), \qquad (66.3)$$

we conclude – after substituting in Eq. 66.3 for a and b from Eqs. 63.7 and 63.8, that

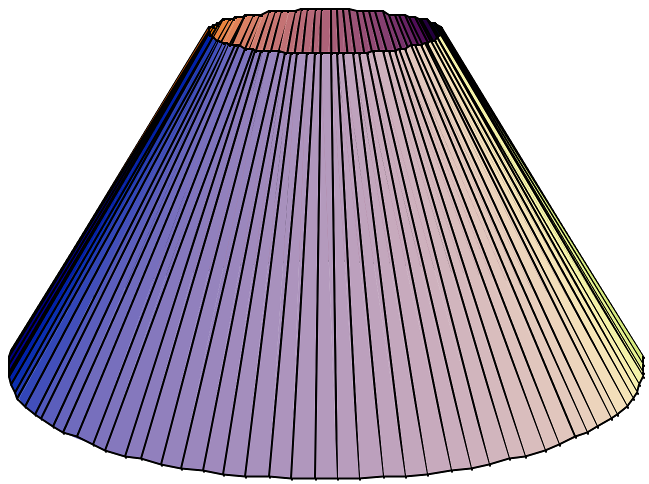
$$A = (n+6) \quad \cot(2\pi / n)/2 \qquad (66.4)$$
$$B = (3n+2) \cot(2\pi / n)/2 \qquad (66.5)$$

From Eqs. 66.1, 66.4, and 66.5, we conclude that

$$A : B : ht = \lim_{n\to\infty} [(n+6)\cot(2p\ /n)/2] : \lim_{n\to\infty} [(3n+2)\ \cot(2p\ /n)/2] : \lim_{n\to\infty}\ [(n\ \text{-}1)\ \cot(2p\ /n)\ ]$$

$$= 1 : 3 : 4 \qquad (66.5)$$

The illustration on the previous page shows a silhouette with these proportions. This asymptotic $_p$ZIGGURAT would be indistinguishable from a truncated conical shell, which is shown below in a perspective view.



Asymptotic *p*-ZIGGURAT

67. Canonical coloring for $p \neq 4$

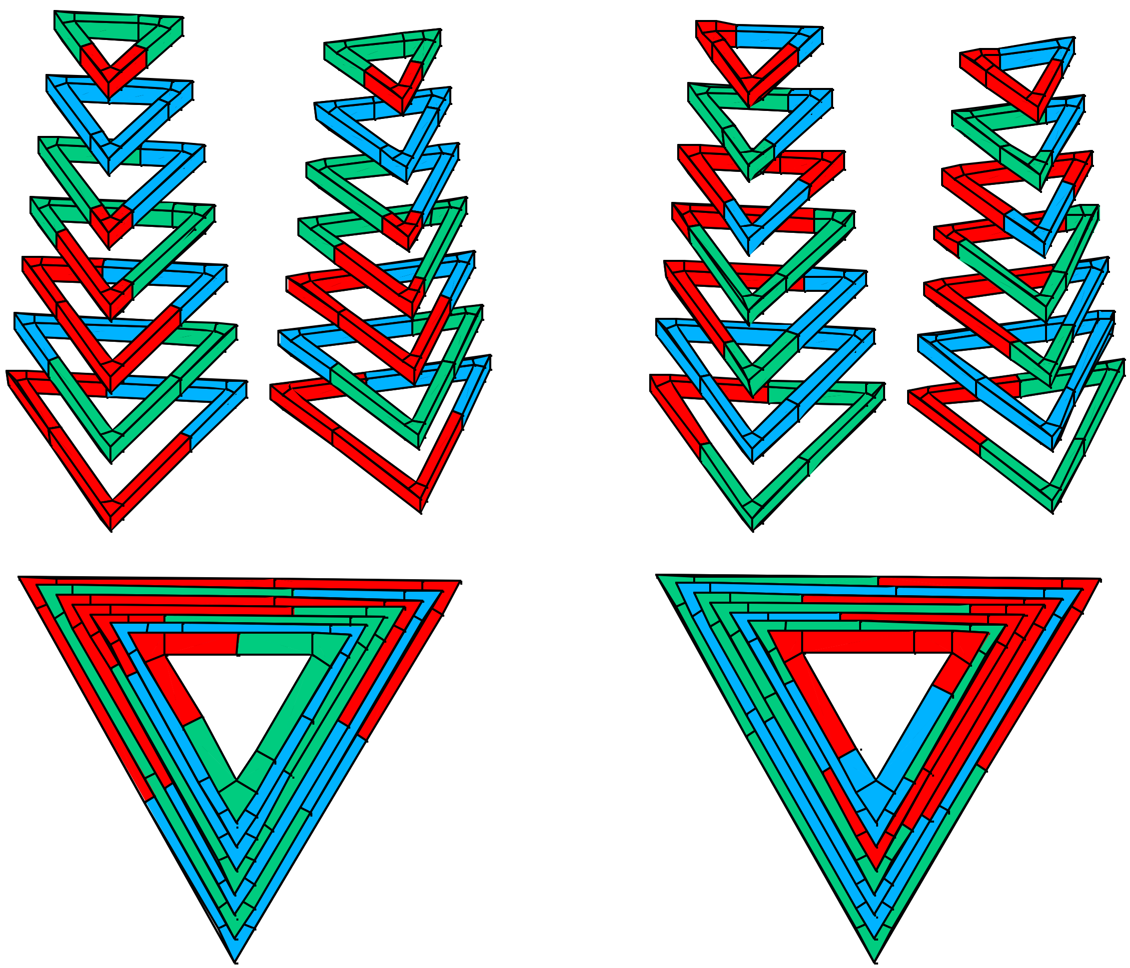
Note: For $p \neq 4$, it is impossible – for either even or odd n – to partition a $_p$LOMINOES set into *convex* subsets. Canonical coloring of subsets is nevertheless defined in the same way as for p=4, by assigning $\lfloor n/2 \rfloor$ colors – identified below by consecutive integers – to the pieces in successive NW-SE diagonal strips of the Triangular Array, starting from upper right and proceeding to lower left (*cf.* p. 5):

even n: 1, 2, …, $\lfloor n/2 \rfloor$-1, $\lfloor n/2 \rfloor$, $\lfloor n/2 \rfloor$-1, $\lfloor n/2 \rfloor$-2, …, 2, 1;
odd n: 1, 2, …, $\lfloor n/2 \rfloor$-1, $\lfloor n/2 \rfloor$; $\lfloor n/2 \rfloor$, $\lfloor n/2 \rfloor$-1, $\lfloor n/2 \rfloor$-2, …, 2, 1.

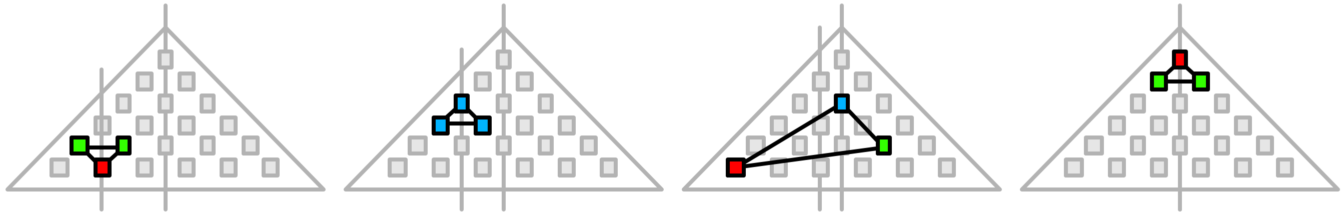
68. The trigonal $_3$ZIGGURAT $_1$[37|6,12]$_1$

Shown below are two dual packings of the trigonal $_3$ZIGGURAT $_1$[37|6,12]$_1$. Solution (a) is on the left; solution (b) is on the right. A full tree search confirms that there are no other packings of this $_3$ZIGGURAT.

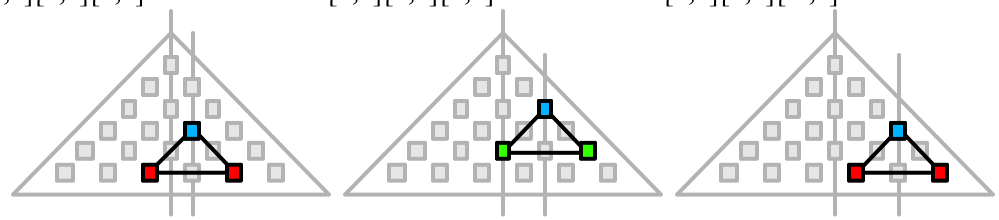
The top-to-bottom color sequence in (a) is the same as the bottom-to-top color sequence in (b). The relation between the color distributions of these two dual packings is a simple consequence of the color symmetry in the canonically colored Triangular Array.



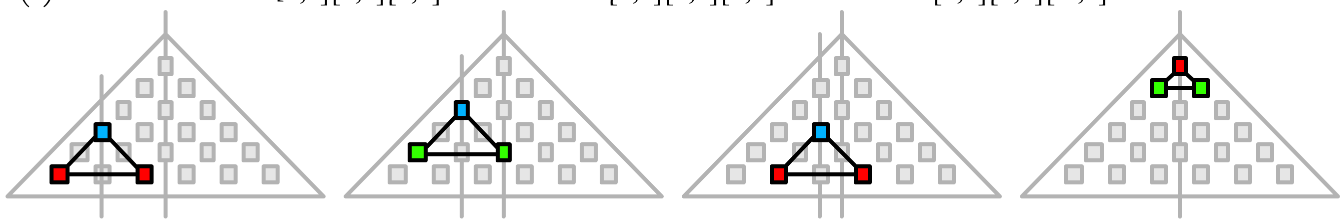
69. The triangular rings of the two dual packings of the trigonal $_3$ZIGGURAT $_1[_37\,|\,6,12]_1$



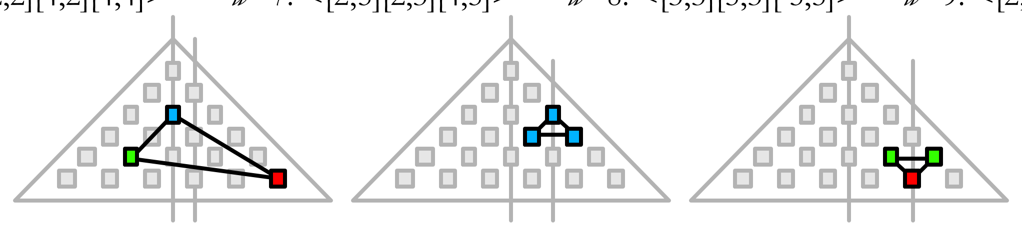
w=6: <[2,3][3,3][3,4]> *w*=7: <[2,5][2,4][3,5]> *w*=8: <[2,2][6,5][3,6]> *w*=9: <[2,7][2,6][3,7]>



solution (a) *w*=10: <[4,4][6,4][6,6]> *w*=11: <[4,5][6,7][4,7]> *w*=12: <[5,5][7,5][7,7]>



w=6: <[2,2][4,2][4,4]> *w*=7: <[2,5][2,3][4,5]> *w*=8: <[3,3][5,3][5,5]> *w*=9: <[2,7][2,6][3,7]>



solution (b) *w*=10: <[3,4][6,3][7,7]> *w*=11: <[4,6][5,7][4,7]> *w*=12: <[5,6][6,6][6,7]>

We conclude from the absence of any systematic pattern in these sets of triangular rings that it is extremely improbable that there exists a generic algorithm, analogous to TOW (*cf.* pp. 36-38), for deriving a packing of a trigonal $_3$ZIGGURAT for any $_3$LOMINOES sets $_3$L*n*.

70. There are no magical shortcuts for packing ZIGGURATS!

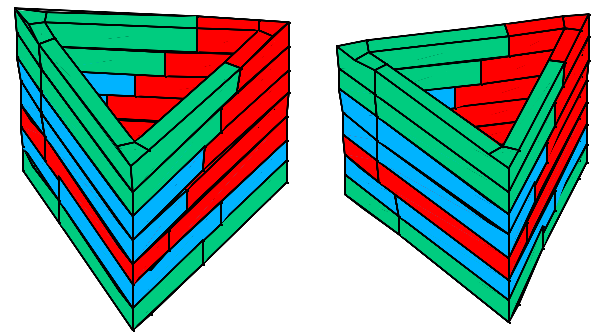
The absence of an orderly pattern in the arrangement of the pieces in any of the 384 packing solutions (*i.e.,* 192 solutions plus their duals) for the ZIGGURAT ${}_1[8|7,13]_1$, together with the evidence cited on p. 105 for ${}_1[{}_37|6,12]_1$, strongly suggest that are no 'hidden secrets' [BCG 2004] lurking here, *i.e.,* there is no special 'generic' pattern for ${}_p$ZIGGURAT packings for *any* value of p (*cf.* p. 86). Trial-and-error – whether by hand or by systematic computer search – seems to be the only way to construct a ${}_p$ZIGGURAT.

Conclusion:

It appears that no generic ${}_p$ZIGGURAT packing algorithm exists for any value of p.

71. The seven triangular rings of the trigonal ${}_3$TOWER T[$_3$7 | 9,9]$_7$

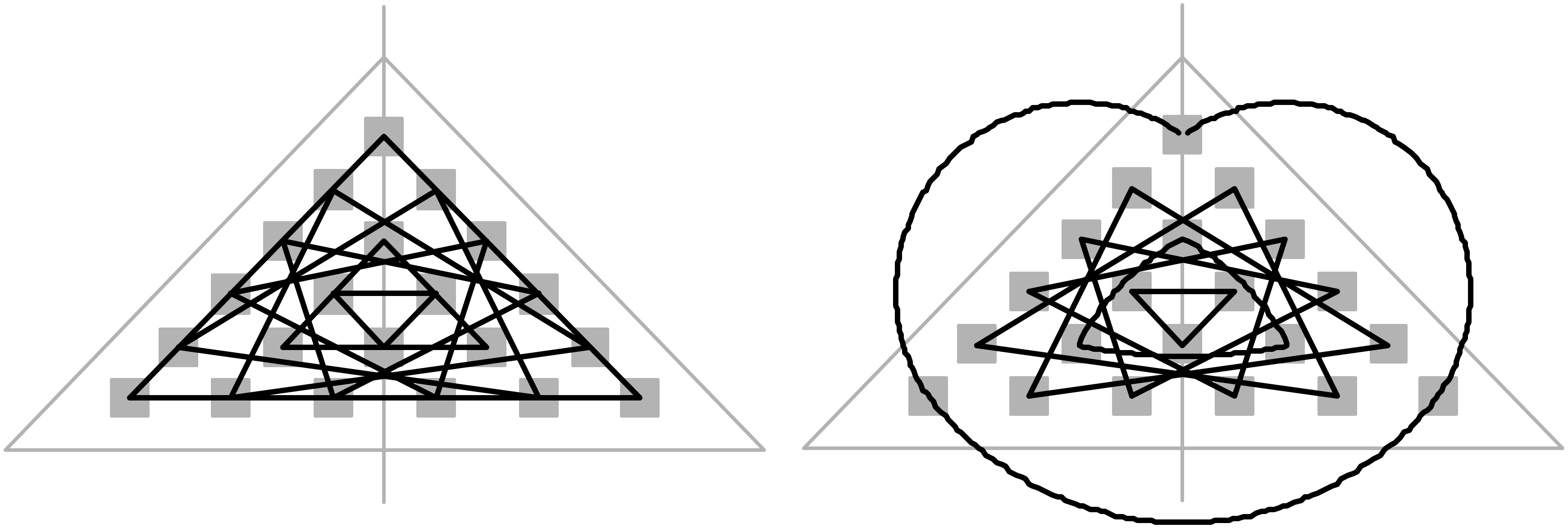
Triangular ring circuit diagrams are shown on pp. 108-110 for the eight solutions of the solitary trigonal ${}_3$TOWER T_1[$_3$7 | 9,9]$_7$. Each of the eight solutions is represented by two graphs. In the first graph, edges are drawn as straight lines, but in the second, some of the edges are curved in order to make the connections clearer. Signatures for these eight solutions are listed on p. 112.



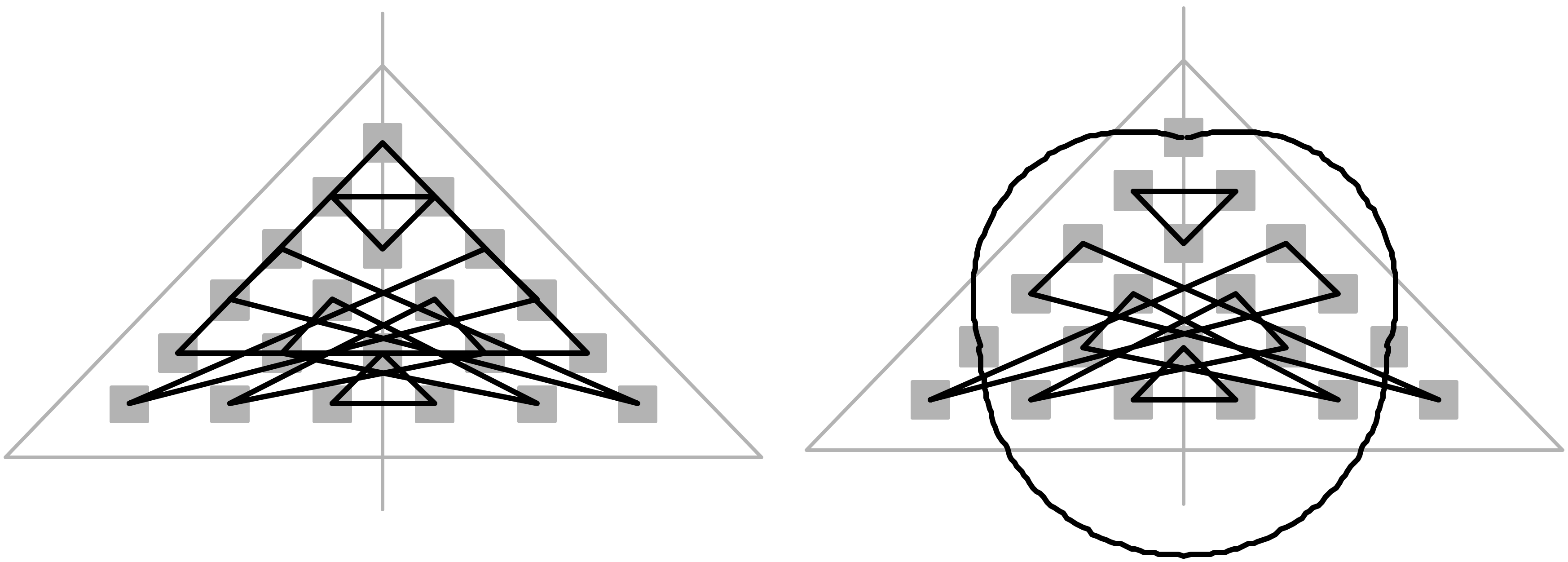
Stereogram of the ${}_3$TOWER T[$_3$7 | 9,9]$_7$

Packing solution no. 1 for the ${}_3$TOWER T_1[$_3$7 | 9,9]$_7$ is derived from a p=3 analog of the TOW algorithm (*cf.* pp. 36-38). The algorithm is slightly simpler for p=3 than for p=4, because of the nice match between the Triangular Array and triangular rings. No simple analog of the TOW algorithm has been found for $p \geq 5$.

A ${}_p$TOWER packing is called self-dual if the set of its p-gonal rings is the same as the set of duals of those rings. Packing solutions 1-6 for T_1[$_3$7 | 9,9]$_7$ are self-dual; in each case, three of the seven rings are self-dual, while the remaining four rings comprise two dual pairs. Every triangular ring in solution 7 is dual to a ring in solution 8.

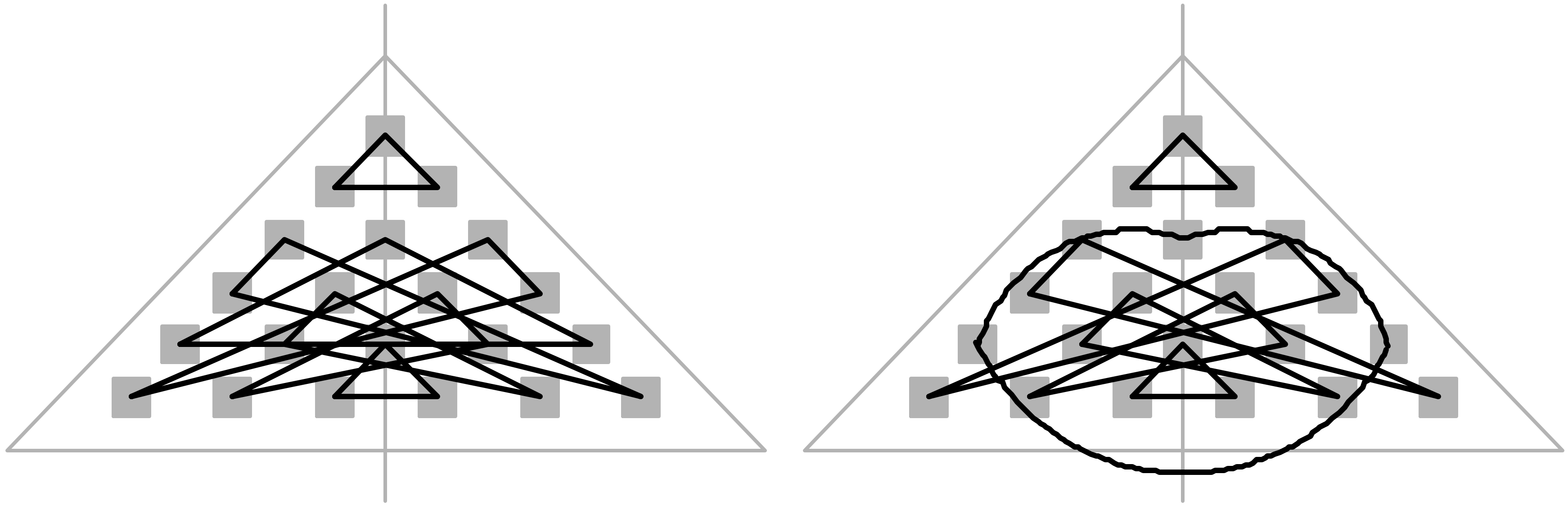


no. 1

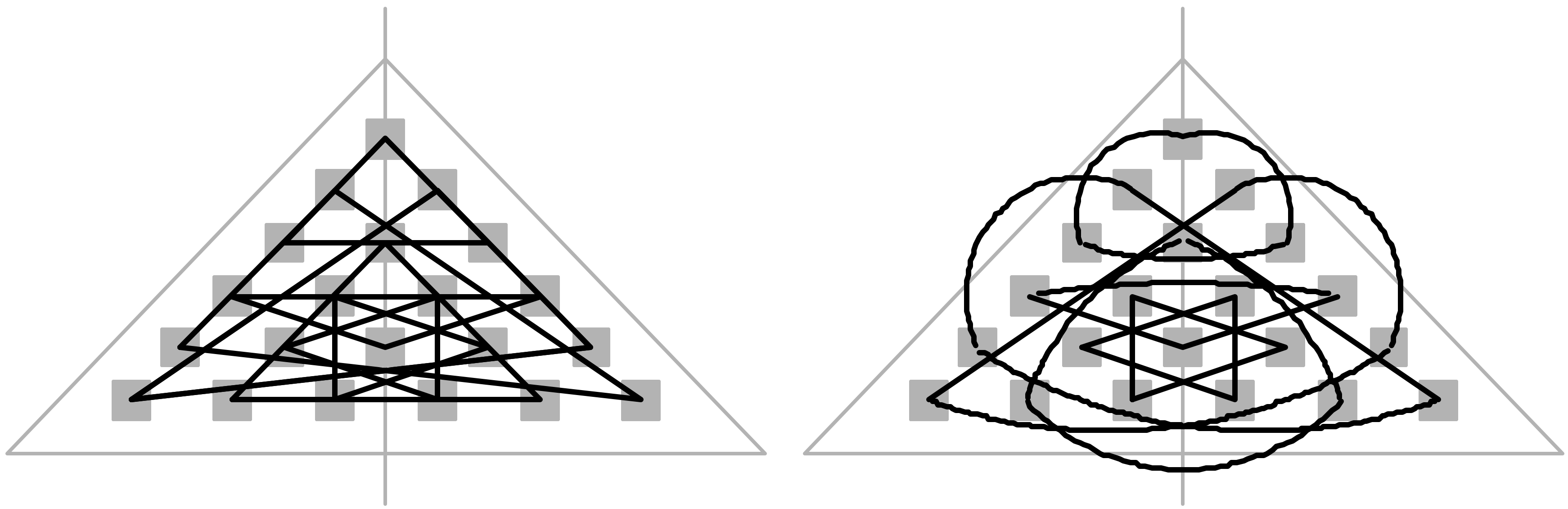


no. 2

Packing solutions 1 and 2 for the trigonal ${}_3$TOWER $T_1[{}_3 7|9,9]_7$

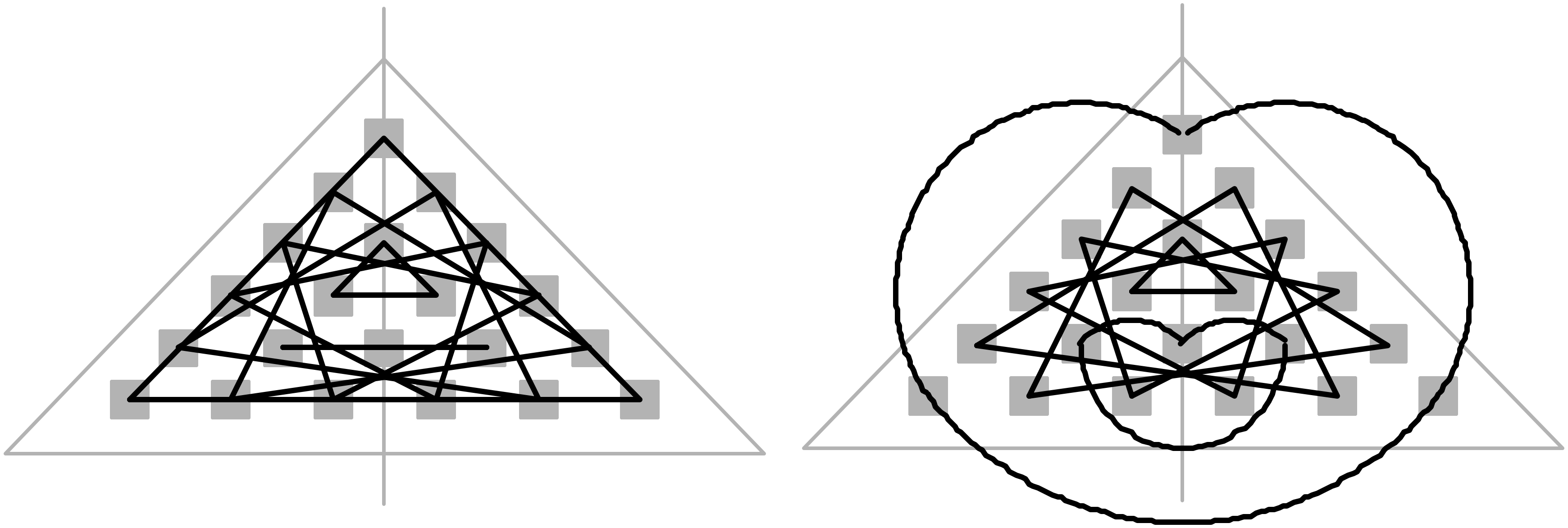


no. 3

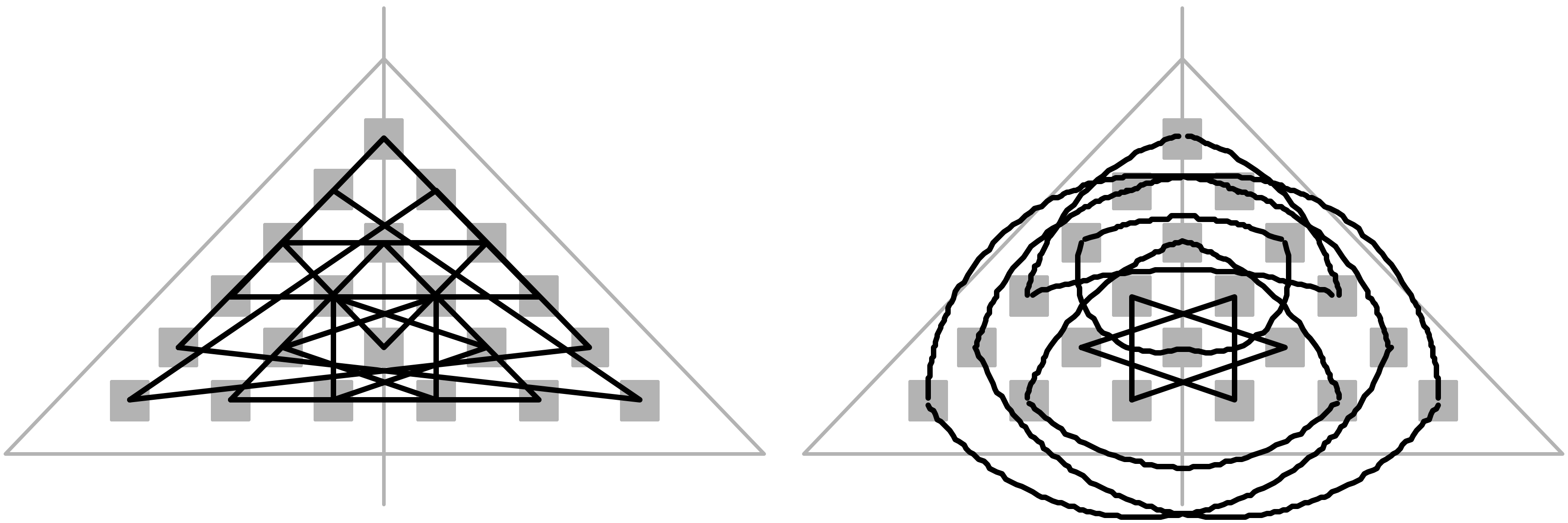


no. 4

Packing solutions 3 and 4 for the trigonal $_3$TOWER $T_1[_37|9,9]_7$

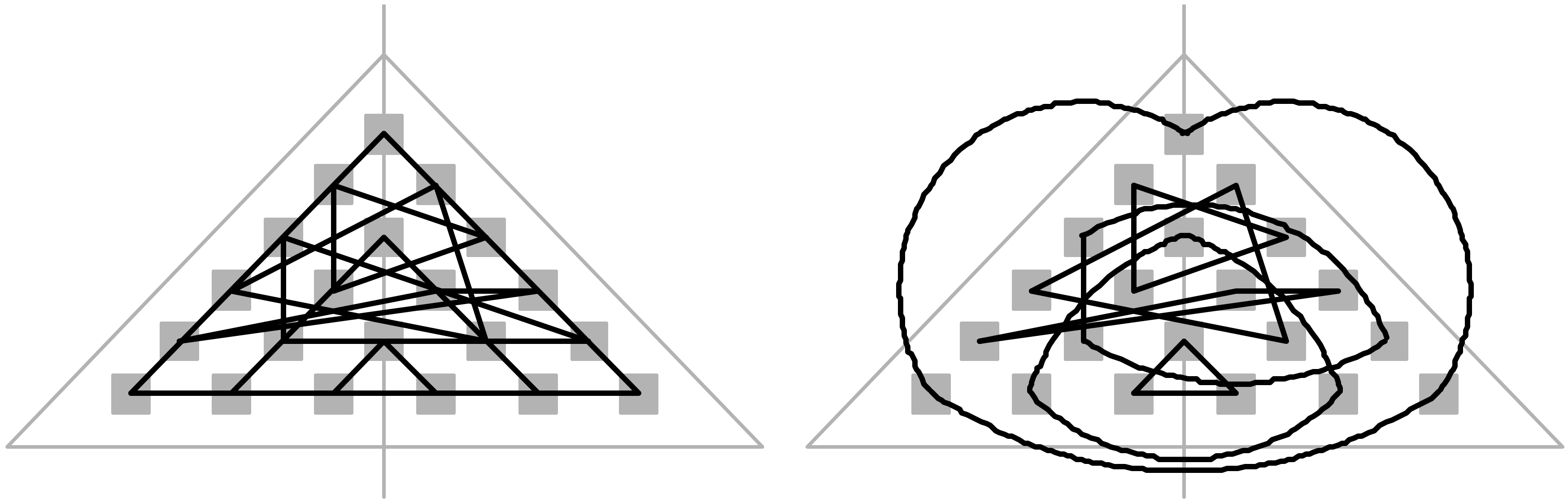


no. 5

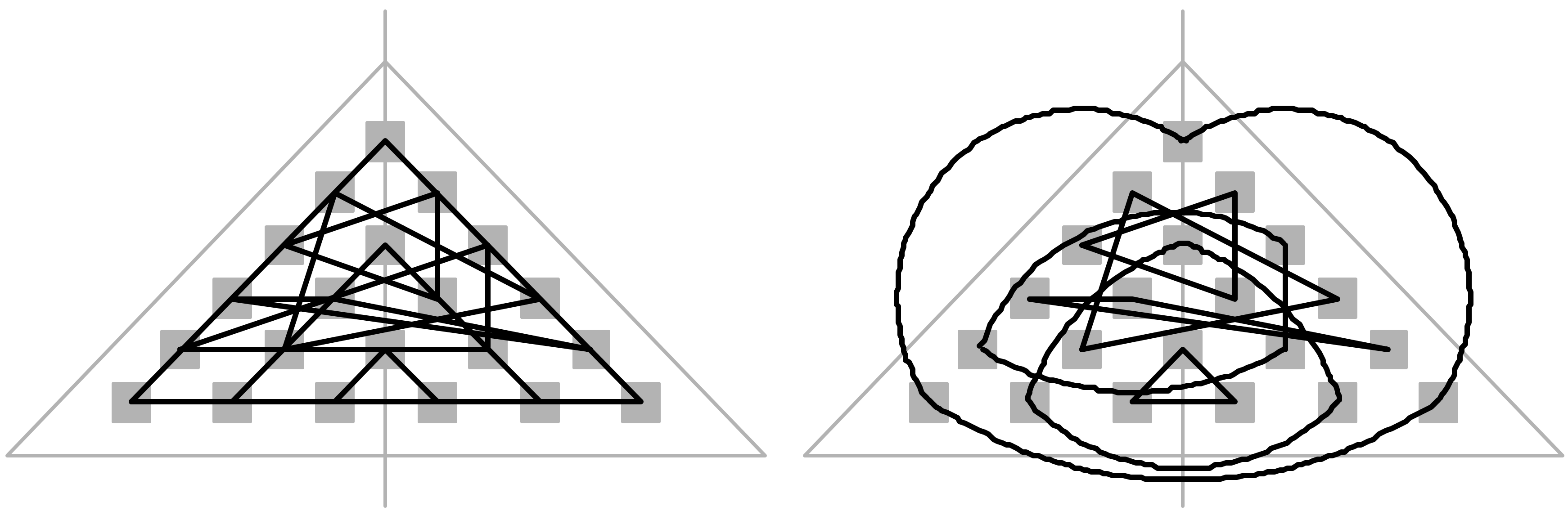


no. 6

Packing solutions 5 and 6 for the trigonal $_{3}$TOWER $T_{1}[{}_{3}7\,|\,9,9]_{7}$



no. 7



no. 8

Packing solutions 7 and 8 for the trigonal ${}_3$TOWER $T_1[{}_37|9,9]_7$

<[2,7][2,2][7,7]>	<[2,6][3,7][2,7]>	<[2,6][3,6][3,7]>	<[2,6][3,7][2,7]>	<[3,5][4,5][4,6]>	<[2,5][4,7][2,7]>	<[2,5][4,6][3,7]>	<[2,5][4,5][4,7]>
<[2,6][3,3][6,7]>	<[4,4][5,4][5,5]>	<[4,4][5,4][5,5]>	<[4,4][5,4][5,5]>	<[4,4][5,2][7,5]>	<[4,4][5,3][6,5]>	<[4,4][5,4][5,5]>	<[4,4][5,3][6,5]>
<[2,5][4,4][5,7]>	<[3,4][5,3][6,6]>	<[3,4][5,3][6,6]>	<[2,6][3,7][2,7]>	<[3,4][5,6][3,6]>	<[3,4][5,5][4,6]>	<[3,4][5,7][2,6]>	<[3,4][5,5][4,6]>
<[2,4][5,5][4,7]>	<[2,4][5,2][7,7]>	<[2,4][5,2][7,7]>	<[4,4][5,4][5,5]>	<[2,4][5,5][4,7]>	<[2,4][5,4][5,7]>	<[2,4][5,3][6,7]>	<[2,4][5,7][2,7]>
<[2,3][6,6][3,7]>	<[3,3][6,6][5,6]>	<[3,3][6,4][5,6]>	<[2,6][3,7][2,7]>	<[3,3][6,2][6,6]>	<[3,3][6,3][6,6]>	<[3,3][6,3][6,6]>	<[3,3][6,3][6,6]>
<[3,6][3,4][5,6]>	<[2,3][6,3][6,7]>	<[2,3][6,7][2,7]>	<[4,4][5,4][5,5]>	<[2,2][6,6][3,7]>	<[2,3][6,2][7,7]>	<[2,3][6,5][4,7]>	<[2,3][6,2][7,7]>
<[3,5][4,5][4,6]>	<[2,2][7,4][5,7]>	<[2,2][7,4][5,7]>	<[2,6][3,7][2,7]>	<[2,2][7,2][7,7]>	<[2,2][7,3][6,7]>	<[2,2][7,2][7,7]>	<[2,2][7,3][6,7]>
no. 1	no. 2	no. 3	no. 4	no. 5	no. 6	no. 7	no. 8

Triangular-ring signatures for the eight packing solutions for the trigonal TOWER $T_1[_37\,|\,9,9]_7$

____

There are two solutions of the trigonal $_3$TOWER $T_1[_36\,|\,8,8]_5$ (plus their duals):

	<[3,4][4,4][4,5]>		<[5,4][4,4][4,3]>
	<[2,4][4,6][2,6]>		<[6,4][4,2][6,2]>
a =	<[3,3][5,3][5,5]>	dual(a) =	<[5,5][3,5][3,3]>
	<[2,3][5,2][6,6]>		<[6,5][3,6][2,2]>
	<[2,2][6,5][3,6]>		<[6,6][2,3][5,2]>

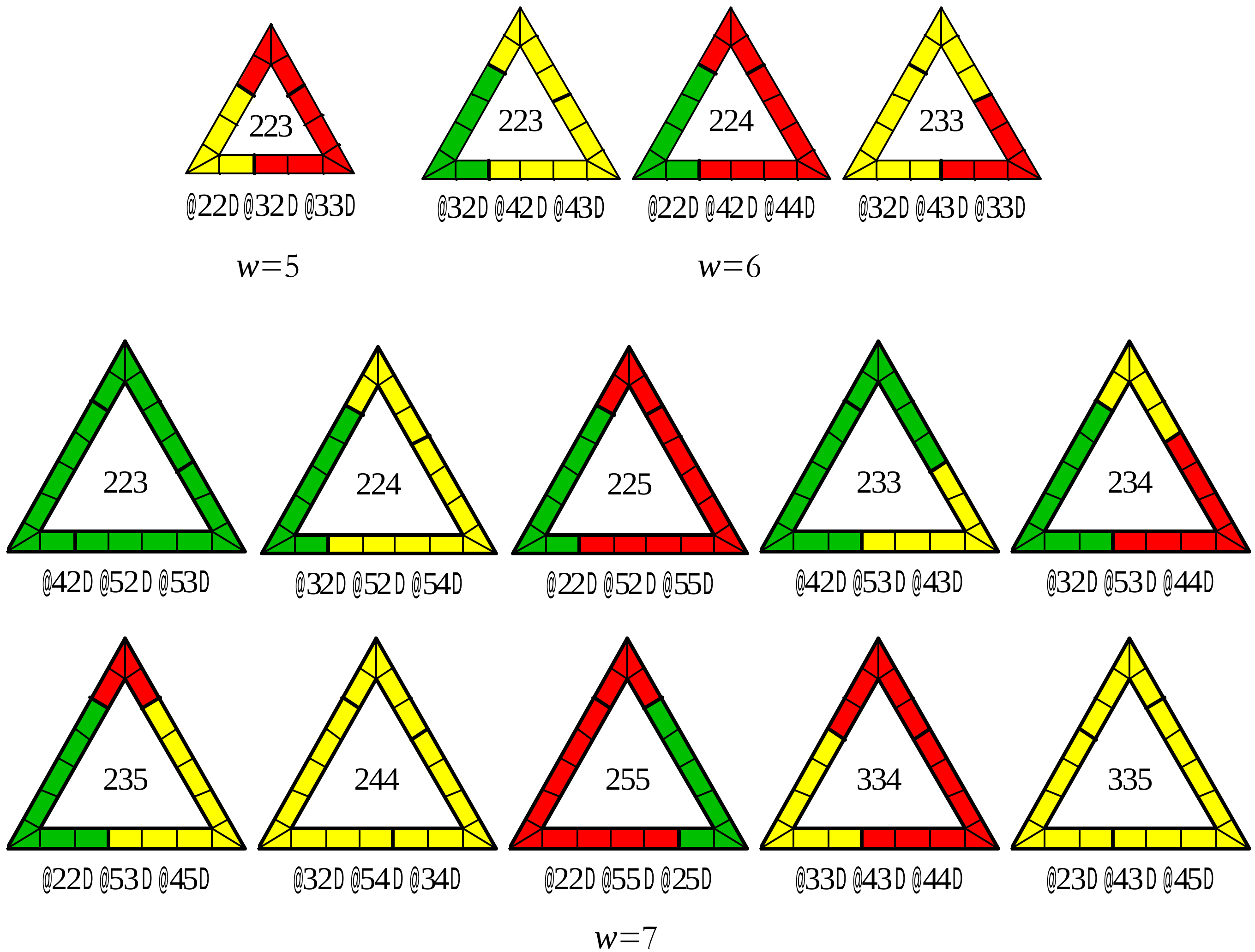
and

	<[3,4][4,5][3,5]>		<[5,4][4,3][5,3]>
	<[2,4][4,4][4,6]>		<[6,4][4,4][4,2]>
b =	<[3,3][5,2][6,5]>	dual (b) =	<[5,5][3,6][2,3]>
	<[2,3][5,5][3,6]>		<[6,5][3,3][5,2]>
	<[2,2][6,2][6,6]>		<[6,6][2,6][2,2]>

72. The tilings of triangular rings for *w*=5, 6, and 7

The 3LOMINOES in the tilings shown below for triangular rings for *w*=5, 6, and 7 belong to a canonically colored L7 set. *M*3(5)=1, *M*3(6)=3, and *M*3(7)=10 (*cf.* p. 97), *i.e.,* there is one ring of width five, three of width six, and ten of width seven.

Each triangular ring tiling is identified by its cut sequence (printed inside each ring) and also by its signature (printed just below each ring). The positions of the cuts in each cut sequence are measured CCW from the corners of the ring, starting from the top of the ring.

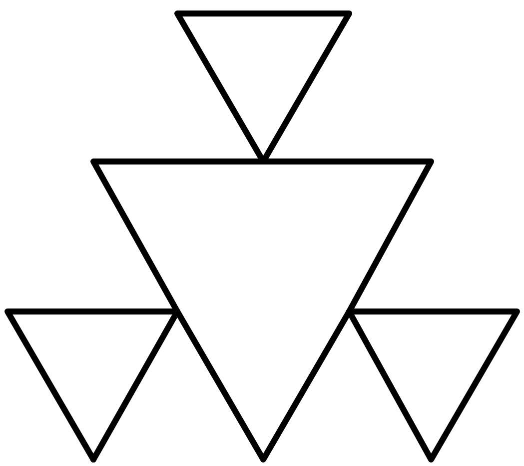
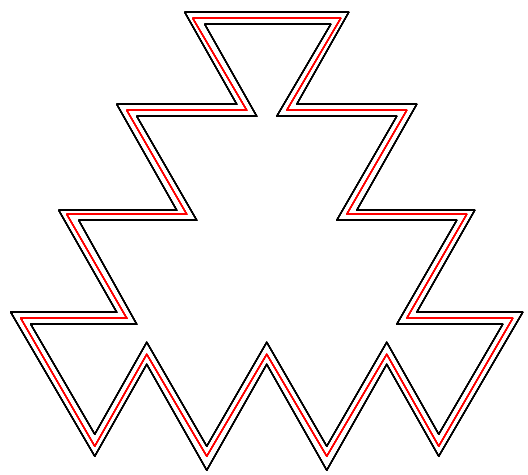


73. More about FILIGREES

A ${}_p$FILIGREE, which is defined for certain values of $p \neq 4$ by analogy with the FILIGREE for p=4 (*cf.* pp. 13, 115), is a closed circuit obtained by folding a ${}_p$SAWTOOTH at p of the junctions between pieces. It is composed of the $N(n)$ ${}_p$LOMINOES of order n=2p or 2p+1 and is required to have Dp symmetry. The smallest *bona fide* examples of ${}_p$FILIGREES occur for p=5 (*cf.* pp. 116-117). Additional examples are shown on pp. 118-123. Spurious examples are shown below and on p. 124.

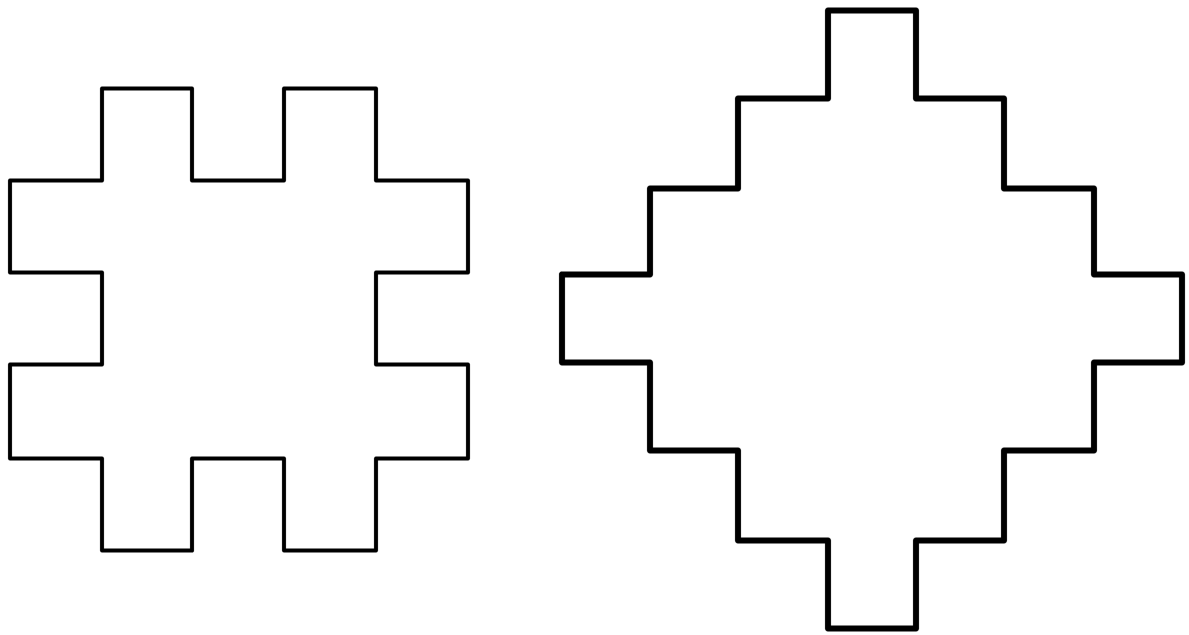
p=3
${}_3$L7; N(7)=21

The fifteen-segment skeleton at below left, which is tiled by the fifteen pieces of ${}_3$L6, is disqualified as a candidate ${}_3$FILIGREE, because it is self-intersecting. At below right is shown an alternative D3-symmetric skeleton, composed of twenty-one segments, that admits a tiling by the twenty-one ${}_3$LOMINOES of ${}_3$L7. The lengths of six of the segments of this skeleton are smaller than the average segment length by one unit, and the lengths of three of its segments are larger than the average segment length by two units. Hence this skeleton does not define a *bona fide* example of a ${}_p$FILIGREE.

p=4
L8; $N(8)$=28

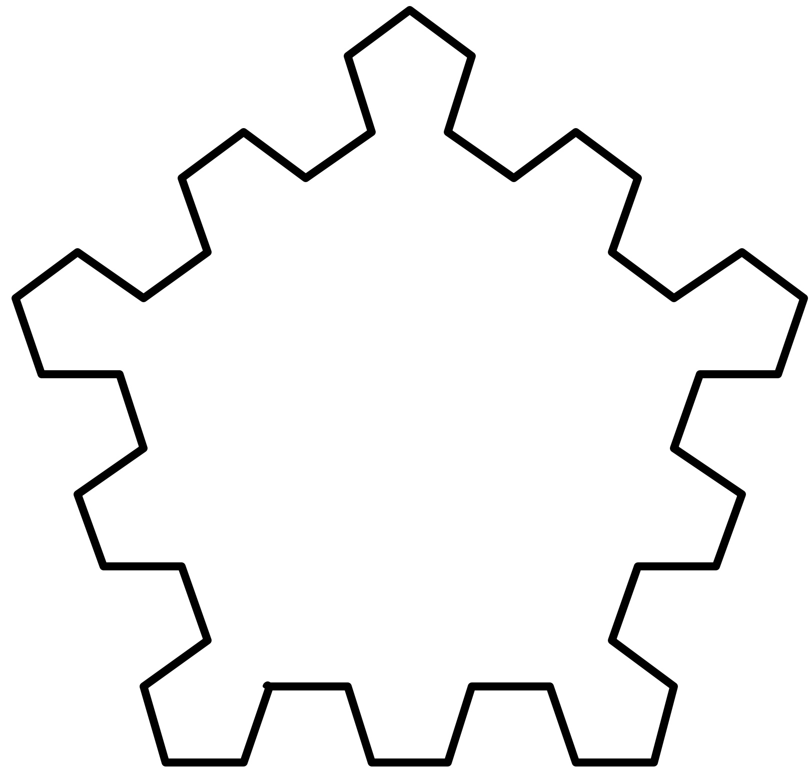
Below are the skeletons of the D4-symmetric L8 FILIGREE (*cf.* p. 13) and EXPANDED FILIGREE (*cf.* p. 20).



p=5

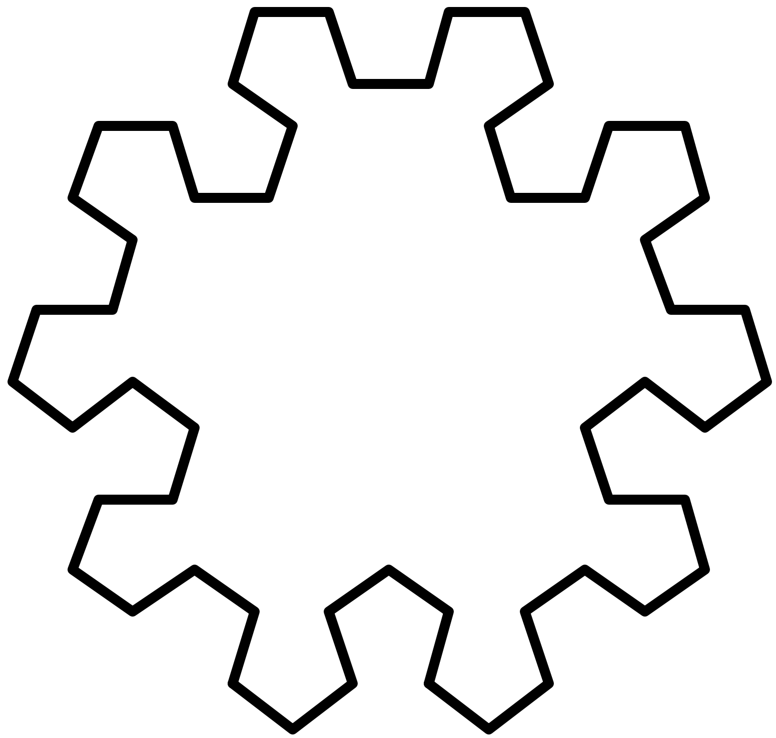
$_5$L10; *N*(10)=45

Below is the skeleton of a $_5$FILIGREE with D5 symmetry that can be tiled by the 45 $_5$LOMINOES of $_5$L10.



p=5
$_5$L11; N(11)=55

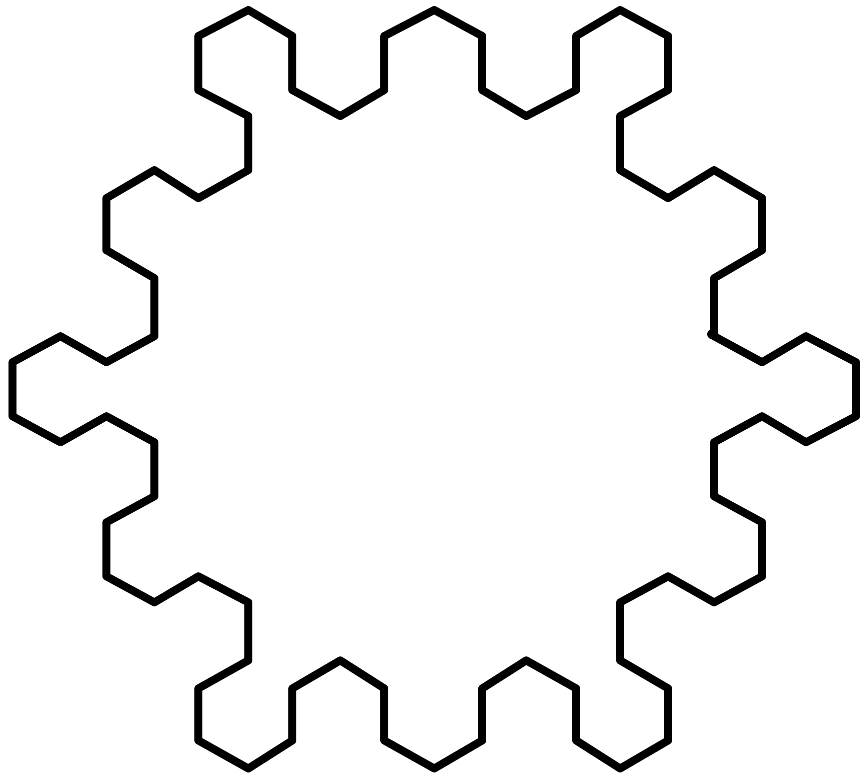
Below is the skeleton of a $_5$FILIGREE with D5 symmetry that can be tiled by the 55 $_p$LOMINOES of $_5$L11.



p=6

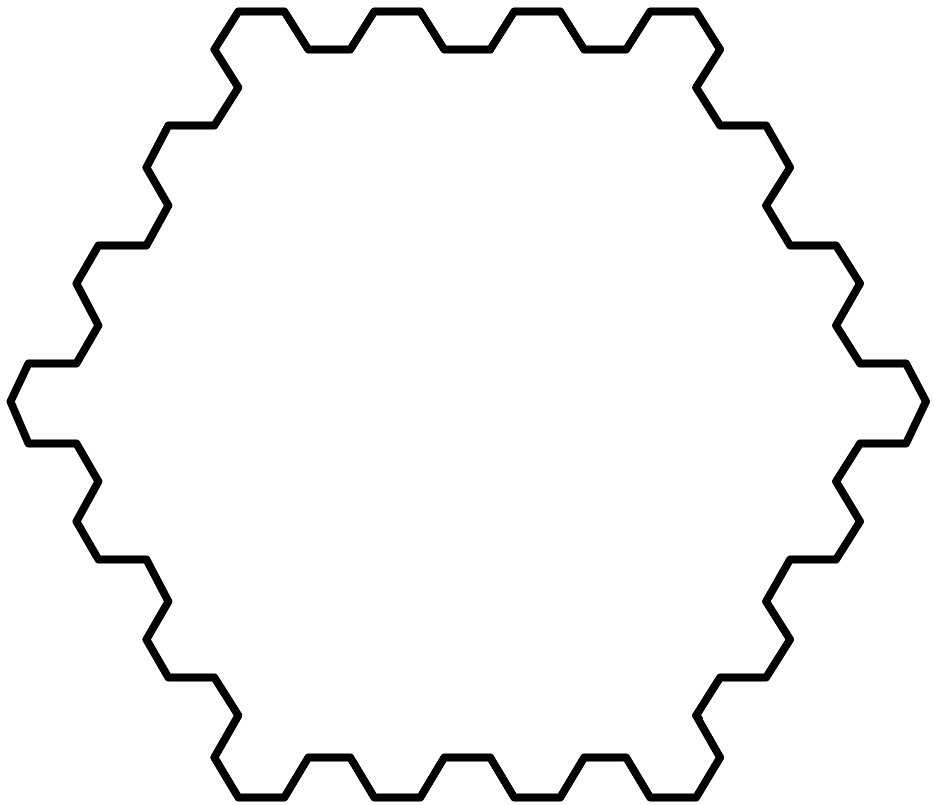
$_6$L13; N(13)=78

Below is the skeleton of a $_6$FILIGREE with D6 symmetry that can be tiled by the 78 $_6$LOMINOES of $_6$L13.



p=6
${}_6$L13; $N(13)$=78

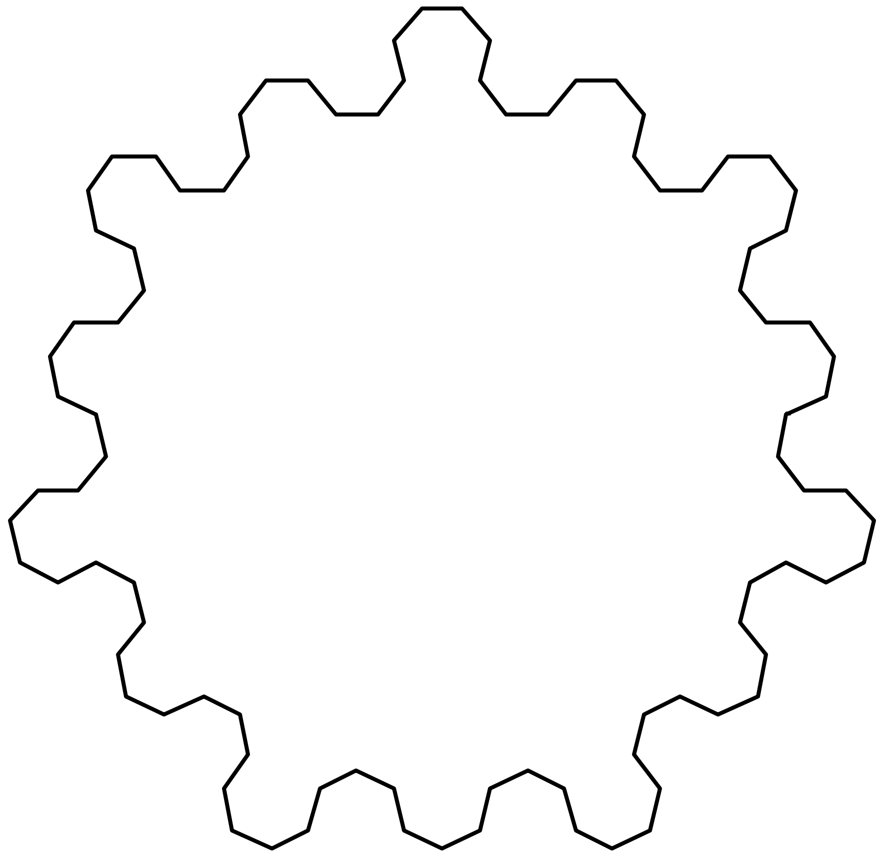
Below is the skeleton of a second ${}_6$FILIGREE with D6 symmetry that can be tiled by the 78 ${}_6$LOMINOES of ${}_6$L13.



p=7

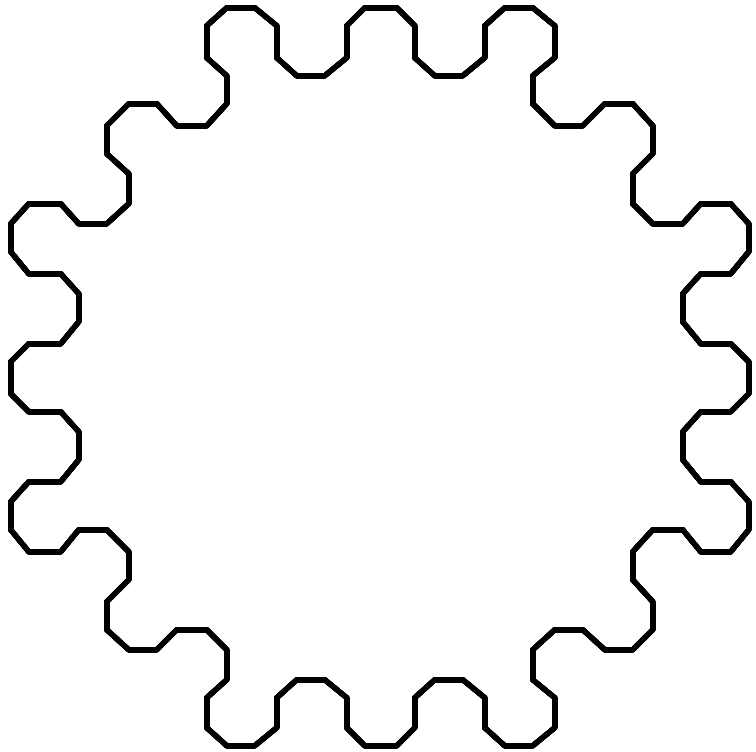
$_7$L14; N(14)=91

Below is the skeleton of a D7 $_7$FILIGREE that can be tiled by the 91 $_7$LOMINOES of $_7$L14.



p=8
$_8$L17; *N*(17)=136

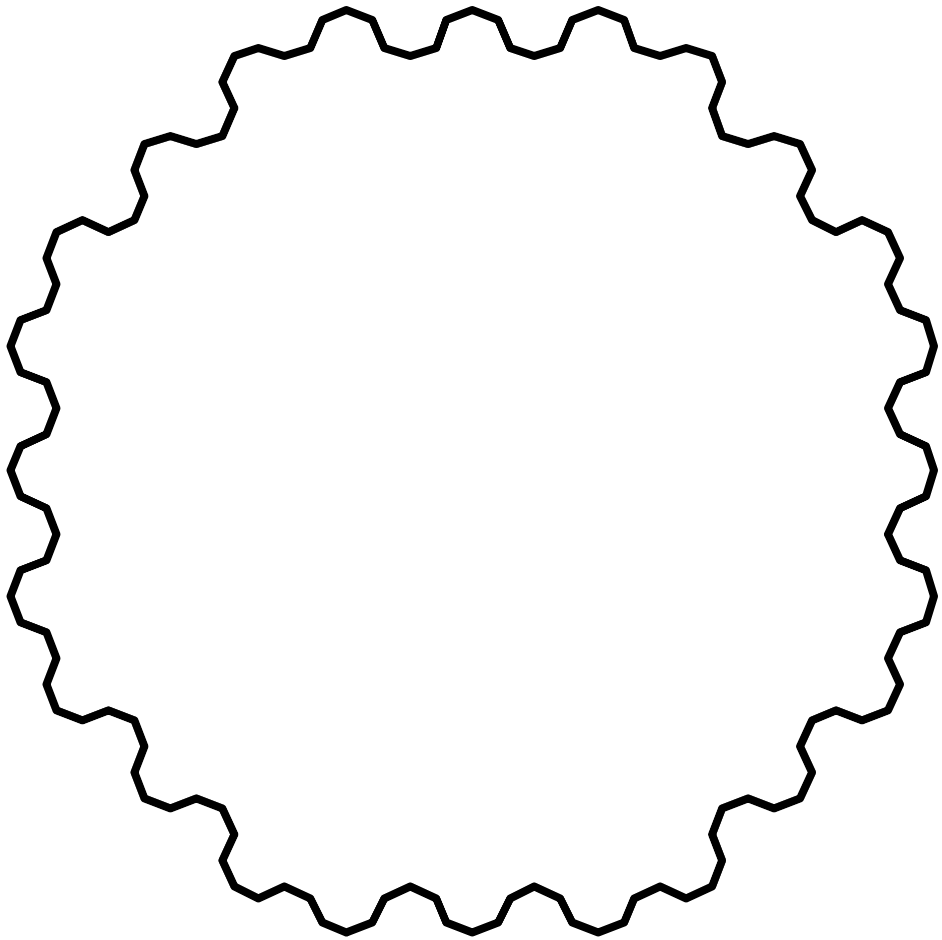
Below is the skeleton of a $_8$FILIGREE with D8 symmetry that can be tiled by the 136 $_8$LOMINOES of $_8$L17.



p=8

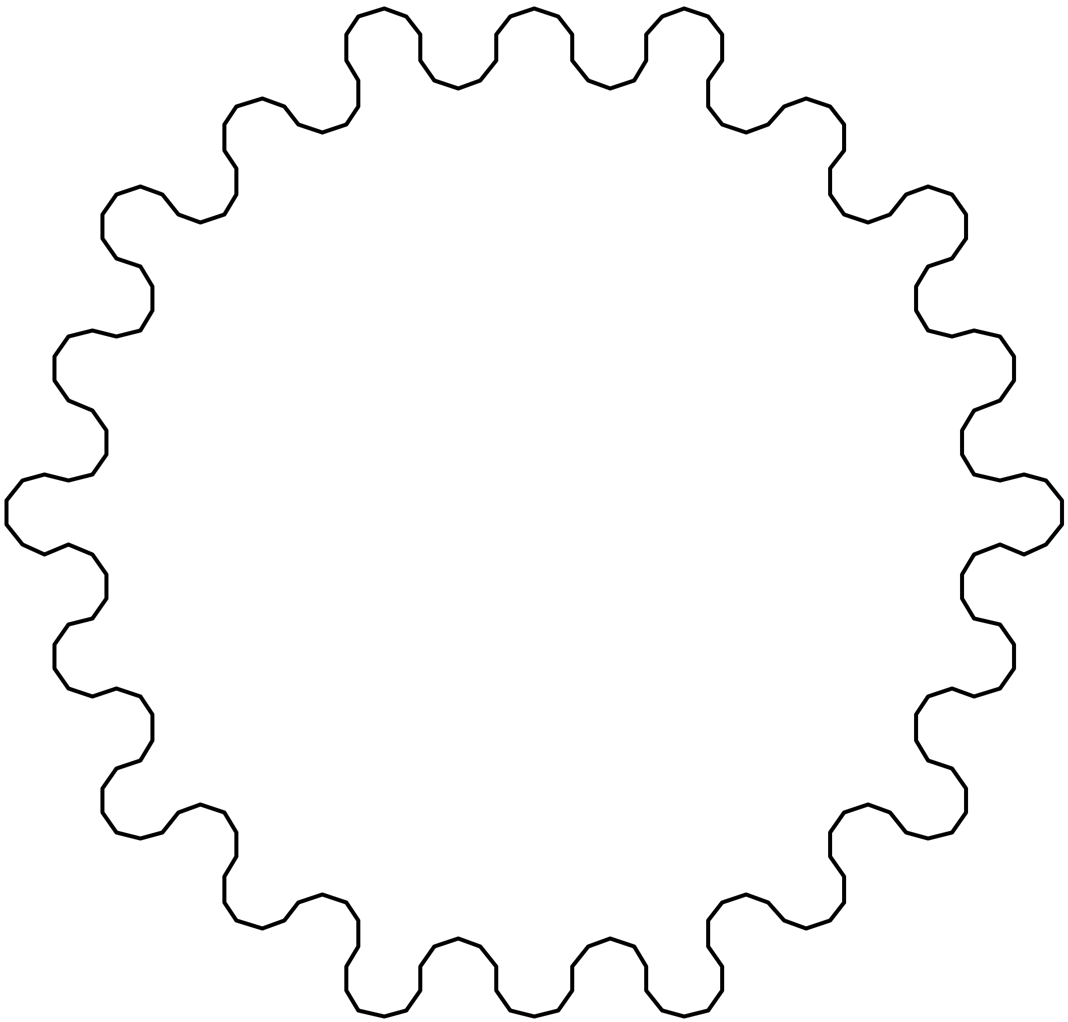
$_8$L17; *N*(17)=136

Below is the skeleton of a second $_8$FILIGREE with D8 symmetry that can be tiled by the 136 $_8$LOMINOES of $_8$L17.



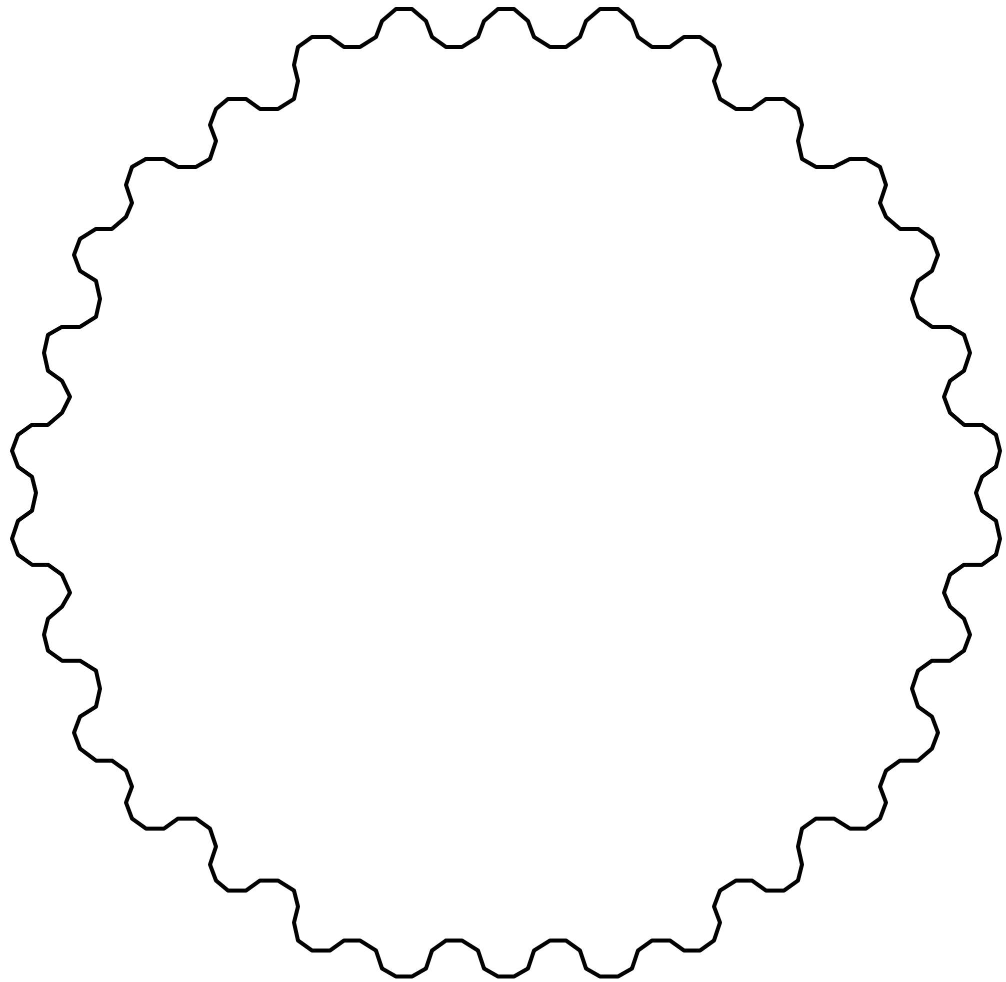
p=10
$_{10}$L21; N(21)=210

Below is the skeleton of a $_{10}$FILIGREE with D10 symmetry that can be tiled by the 210 $_{10}$LOMINOES of $_{10}$L21.



p=10
$_{10}$L21; N(21)=210

Below is the skeleton of a pseudo-$_{10}$FILIGREE. It would require 230 $_{10}$LOMINOES – twenty more than the number of $_{10}$LOMINOES in $_{10}$L21 and one less than the number of $_{11}$LOMINOES in $_{11}$L22. What collection of pieces could you use to tile it?



Exercises

1. Find a SAWTOOTH tiling for L8 or L8† that is a four-color map-coloring.

2. There are sixty-two ways to choose four LOMINOES of L8† to make a *pied* 4-ring, *i.e.,* a 4-ring in which no two LOMINOES are of the same color. Construct an augmented TOWER $T_1[8^\dagger | 10,10]_8$ (*cf.* p. 33) composed of eight such 4-rings. (There are 800 solutions.)

3. Let us define SQL-n, the *square set,* as a set of $(n-1)^2$ LOMINOES based on the Square Array of integer pairs (i,j) $(2 \leq i,j \leq n)$. SQL-n contains two specimens of every LOMINO $[i,j]$ in Ln for which $i \neq j$ and one specimen of each of the n-1 'twin' LOMINOES $[i,i]$ in subset 1 of Ln.

(a) Prove that the area $A^*(n)$ of SQL-n is equal to $(n-1)^2(n+1)$.

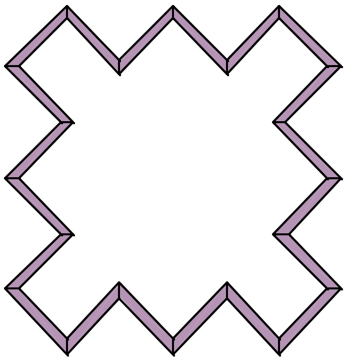
(b) Prove that $A^*(n)$ is a perfect square for $n=k(k+2)$ $(k=1,2,\ldots)$.

(c) Find a tiling of the 21 x 21 SQUARE arena by the forty-nine pieces of SQL-8.

(d) Using the sixteen pieces of SQL-5, find a packing of the 4-story variant ZIGGURAT composed of 4-rings of ringwidths 6, 7, 7, and 8.

4. Transform Greg Martin's sequence (let's call it 'GMS') for the L8 SAWTOOTH (*cf.* pp. 9-10) into a recipe for tiling the solitary augmented TOWER $T_1[8^\dagger|10,10]_8$ (*cf.* p. 33) composed of eight *self-dual* 4-rings. (Hint: First insert the four extra LOMINOES of L8† into GMS. Next remove the four self-dual LOMINOES from GMS and insert them into new positions in GMS.) Does this recipe yield TOWERS for any other augmented sets?

5. Using five L8 sets, construct a seven-floor 20-ring Toltec ZIGGURAT (*cf.* pp. 43-44).



Uniform Toltec 20-ring

6. For p=8, Eqs. 63.7 and 63.8 define the ringwidths of the largest and smallest octagonal rings of the solitary regular octagonal $_8$ZIGGURAT $_1[_816|11,25]_1$, for which it is conjectured there exists a packing. If – as conjectured on p. 54 – packings also exist for the pair of $_4$ZIGGURATS of the medial $_4$ZIGGURAT COMPLEX $_1[_416|11,25]_2$, the total number of packings of such pairs would almost certainly be significantly smaller than the number of packings of $_1[_816|11,25]_1$. Explain why this is likely to be true, and explain also why the solitary octagonal analog $_1[_816|8,24]_1$ of the *irregular* $_4$ZIGGURAT $_1[_416|8,24]_2$ cannot possibly have a packing.

7. Prove that
   (a) for odd n, the $(n-1)/2$ pronic subsets of L$n^\dagger$ tile $(n-1)/2$ rectangles with proportions $1 \times n \times (n+1)$;
   (b) for even n, the $n/2$ pronic subsets of L$n^\dagger$ tile $n/2$ rectangles with proportions $1 \times n \times (n+1)$ (*cf.* p. 1).

8. Since the area of one L8 pronic subset is 72, two such subsets have the area of a 12 x 12 SQUARE (*cf.* p. 75). Find a tiling of the 12 x 12 square by subset pairs 2-3 and 3-4. It is conjectured that none of the other four pairs of subsets tile this square, although 1-3 and 2-4 almost succeed!

9. A perfect number P is one which is equal to the sum of its proper divisors, *i.e.,* all of its divisors including unity but not including P itself. For every known example of a perfect number P, there exists a standard set Ln that contains P LOMINOES. For which one of the known perfect numbers is it impossible to construct a solitary ZIGGURAT from the pieces of the corresponding standard set?

10. What is the minimum value of n for which the set $_9$Ln will admit a tiling of a $_9$FILIGREE with D9 symmetry?

11. Explain why it is possible for TOWERS, but not ZIGGURATS, to admit self-dual packings.

12. Derive the TOW-type algorithm for p=3. Is there an algorithm for assembling $_p$TOWERS for $p \geq 5$? (The answer is unknown.)

13. Which of the eight packing solutions for the trigonal TOWER $T_1[_37\,|\,9,9]_7$ on pp. 108-112 is derived from the TOW algorithm for p=3?

14. The packing solution on p. 91 for the trigonal $_3$TOWER $T_1[_310\,|\,12,12]_{15}$ is *not* derived from the TOW algorithm for p=3. Derive the TOW-based packing solution for this $_3$TOWER.

15. Prove that if a set of $_p$LOMINOES admits a packing of a solitary $_p$TOWER in which the number of floors is even, it does not admit a packing of a solitary $_p$ZIGGURAT.

16. $_p$LOMINOES sets for $n \gg 8$ are impractical as physical puzzles both because the number of pieces is inconveniently large and also because it is difficult to distinguish the larger pieces from one another. Nevertheless we would like to identify the values of n for which $_p$LOMINOES sets define solitary $_p$TOWER or $_p$ZIGGURAT candidates.

Let us call a $_p$TOWER or $_p$ZIGGURAT *basic* if $p=\lfloor n/2 \rfloor$ and *variant* if $p<\lfloor n/2 \rfloor$ (*cf.* p. 91).

(a) Why are there no variant $_p$TOWERS for n<10?

(b) List all possible variant $_p$TOWER candidates for $n \le 25$. (Hint: First identify the prime divisors d of $\binom{n}{2}$ in the interval $3 \le d < \lfloor n/2 \rfloor$.)

(c) For which variant $_p$TOWER candidates is there a $_p$ZIGGURAT counterpart (*cf.* p. 56, 65-66)? (Note that for n=12, for example, the number of floors in the variant $_3$TOWER $T_1[_312\,|\,14,14]_{22}$ is even. For n=16, the number of floors in the variant $_p$TOWERS for p=3, 4, and 5 is also even. But the number of floors is odd in the variant $_p$TOWERS $T_1[_315\,|\,17,17]_{35}$ and $T_1[_515\,|\,17,17]_{21}$.)

17. As discussed on pp. 68 and 75, if LOMINOES are regarded as flat tiles, the total area $A(n)$ of the standard set Ln is $n(n^2-1)/2$ and the total area $A^\dagger(n)$ of the augmented set L$n^\dagger$ is $n^2(n+1)/2$. The thirty-two pieces of L$8^\dagger$ (total area=288) can be arranged to tile two congruent squares with sides of length twelve (*cf.* p. 75).

(a) Prove that for $n=8\binom{k+1}{2}$ (k=1, 2, 3, …),

$$A^\dagger(n)/2 = (12\,s\,(k))^2,$$

where

$$s\,(k) = 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Consequently each augmented set L$n^\dagger$ (n=8, 24, 48, 80, 120, …) is a plausible candidate for tiling two congruent squares.

(b) Prove that in any tiling of two congruent squares by L$n^\dagger$, it is impossible for all of the pronic subsets of L$n^\dagger$ to be embedded.

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AFTERWORD

The sequence of values of ${}_3M_{odd}[w]$ for odd w (*cf.* p. 97) is identical to the sequence A000447 in Neil J. A. Sloane's On-Line Encyclopedia of Integer Sequences (http://www.research.att.com/~njas/sequences/). Among the objects cited there as defining the sequence A000447 are the following:

(1) number of standard (Young) tableaux of shape ($2n$-1,1,1,1) ($n \geq 1$) (Emeric Deutsch);
(2) 4 times the variance of the area under an n step random walk (Henry Bottomley);
(3) polyhedral figurate numbers for structured octagonal diamond, with vertex structure 9 (James Record).

No match is found in Sloane's On-Line Encyclopedia for ${}_3M_{even}[w]$, ${}_3M[w]$, or the multiplicity sequences for p=4 and 5.

The sequence defined by ${}_3Z[w]$ (*cf.* p. 97) is the sequence of tetrahedral numbers [CONGUY 1996, pp. 44-46], which is listed in Sloane's On-Line Encyclopedia as A000292.

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