These informal remarks on K-patterns were written as a kind of interim progress report in the course of preparing a book on the subject, partly in order to share some of my principal results with others and partly to discover for myself which questions appear to show promise for further development. If the organization of the material on color reflection symmetry seems especially diffuse, it is perhaps because it was written last during the few days it was being worked out (and tested).

I have scarcely mentioned what was in retrospect the most difficult part to make come out right--the calculation of period and symmetry for all <n, s> parameter sets, for power residues with $\alpha \leq 7$ (§III. A, p. 8). But this summary was not intended to be exhaustive! In any case, I will try to cover this topic adequately in the forthcoming book.

I welcome any comments, including improvements, extensions, and of course identification of errors.

Alan Schoen

ILLUSTRATIONS

page	n	s	jo	α	sym
Cover	29^3	14	2	29	L
i	2001	43	1	3	d_6
ii	25	5/7	1/7	3	d_{10}
iii	3515	8	1	5	d_5
iv	4123	6	1	7	d_7
v-a	23^3	22	1	23	L
v-b	23^3	20	1	23	L
v-c	29^3	14	1	29	L
v-d	23^3	8	1	23	L
v-e	29^3	8	2	29	L
vi-a	31^3	4	1	31	L
vi-b	31^3	2	1	31	L
vii	37^3	1	1	37	d_2
viii	23^2	23/4	1/4	3	d_{23}
ix	13^2	13/4	1/2	3	d_{13}

The rounded character of most of the K-patterns shown here is the result of:

(a) dividing each calculated unit vector into two collinear half-unit vectors which are calculated but not plotted; and

(b) constructing the pattern by joining the midpoints of all consecutive half-unit vectors.

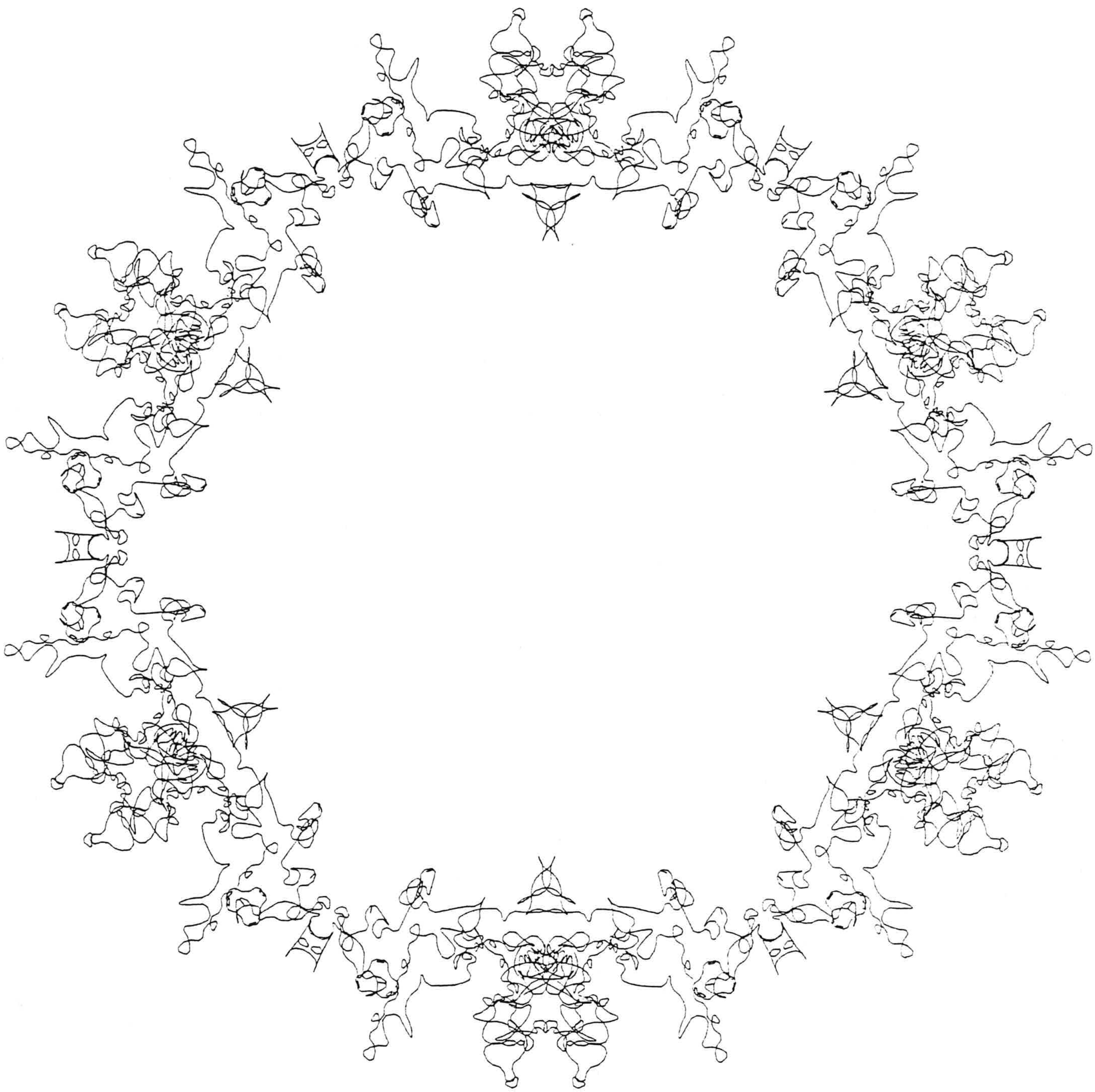
ADDITIONAL ILLUSTRATIONS

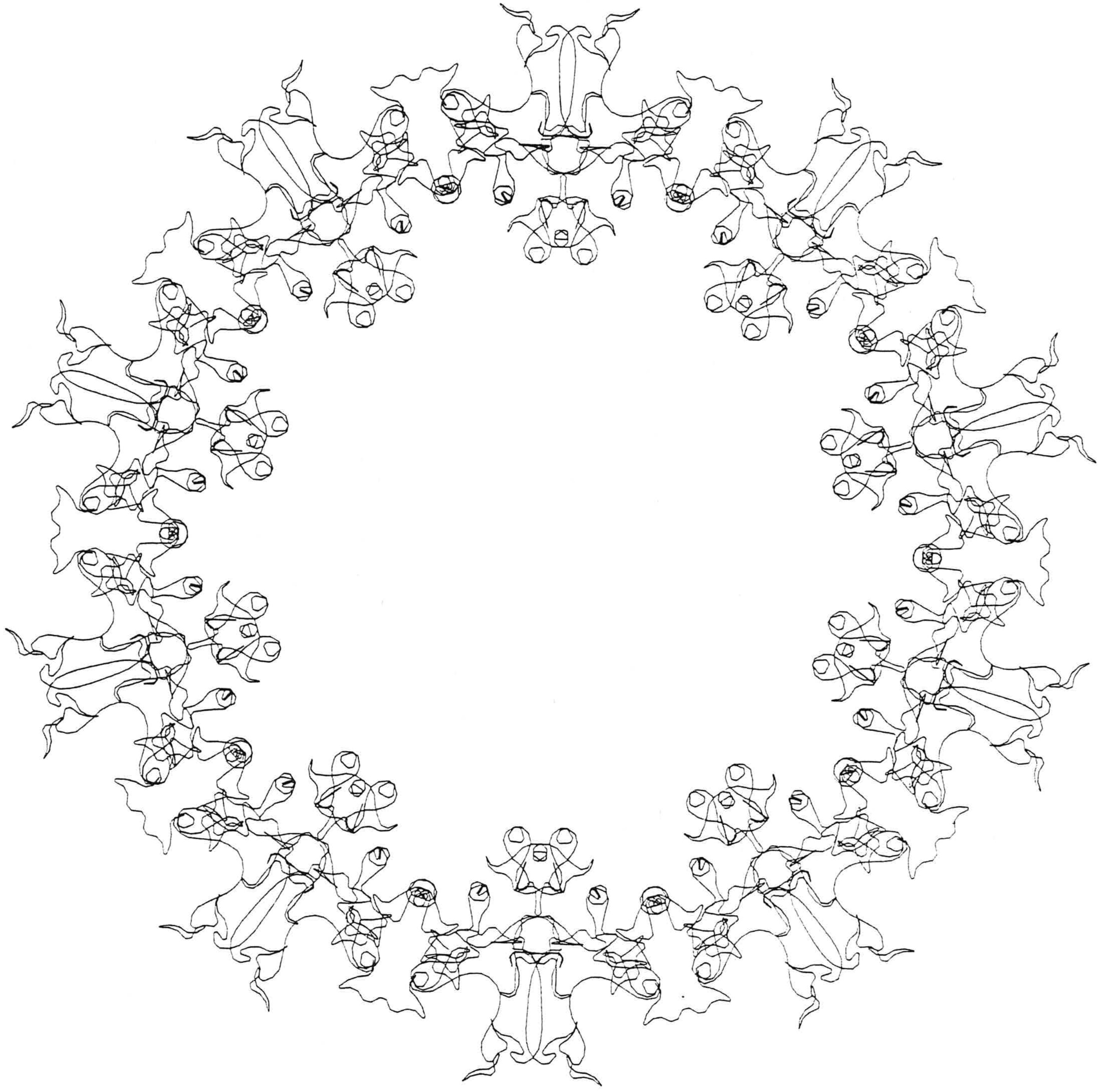
figure	n	s	j_0	α	sym
x	$5 \cdot 19 \cdot 37 = 3515$	36	1	5	d_5
xi	"	12	1	5	d_5
xii	"	24	1	5	d_5
xiii	$37^3 = 50,653$	2	0	37	L
xiv	"	4	0	37	L
xv	"	6	0	37	L
xvi	"	108	0	37	L
xvii	"	110	0	37	L
xviii	"	112	0	37	L
xix*	$5 \cdot 11 = 55$	1 - 54	1	3	various
xx*	$5 \cdot 13 = 65$	1 - 64	1	3	various
xxi*	$5 \cdot 17 = 85$	1 - 50	1	3	various

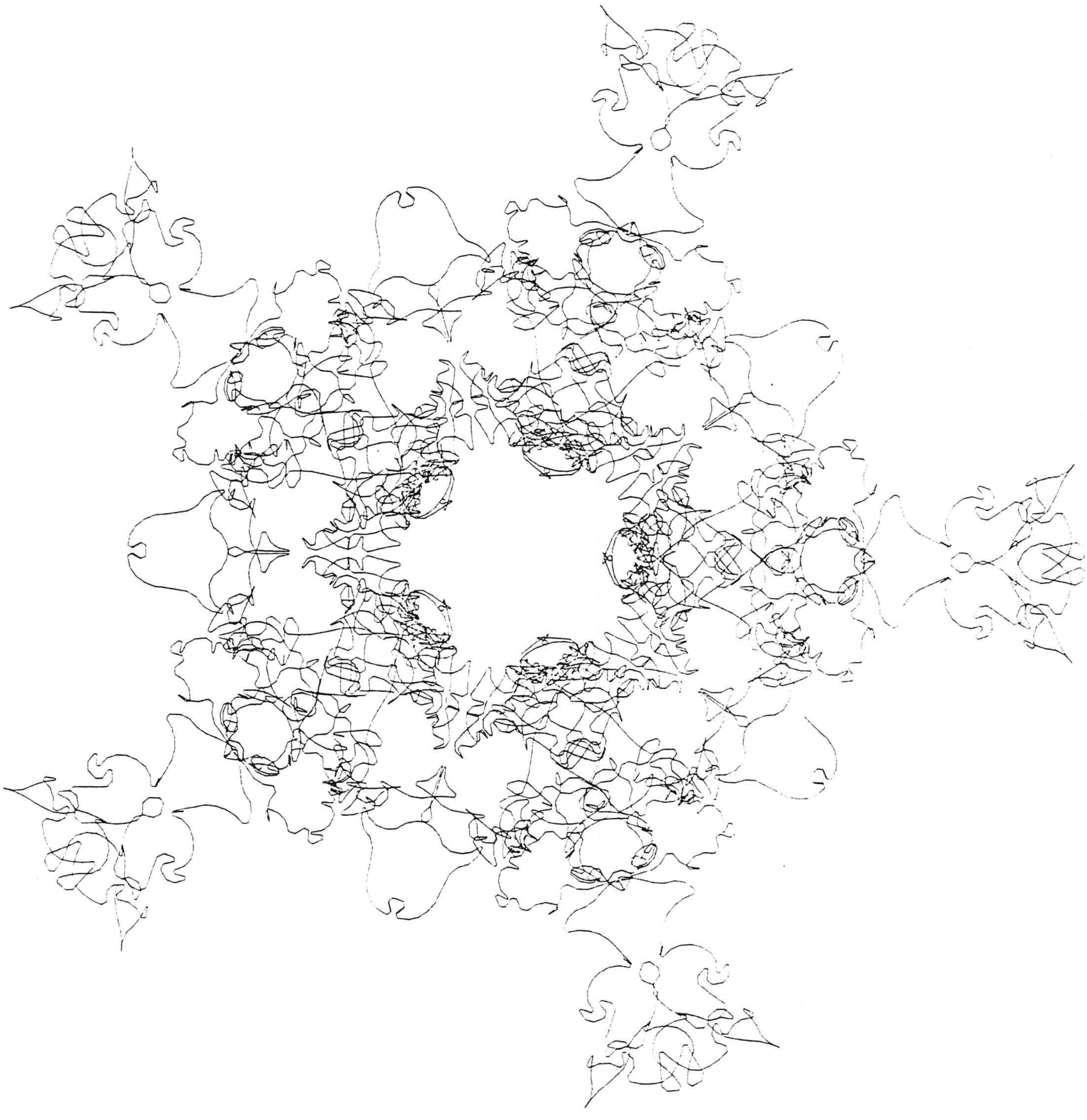
* Pattern with step value = 1 is in first row, first column; pattern with step value = 2 is in second row, first column; etc.

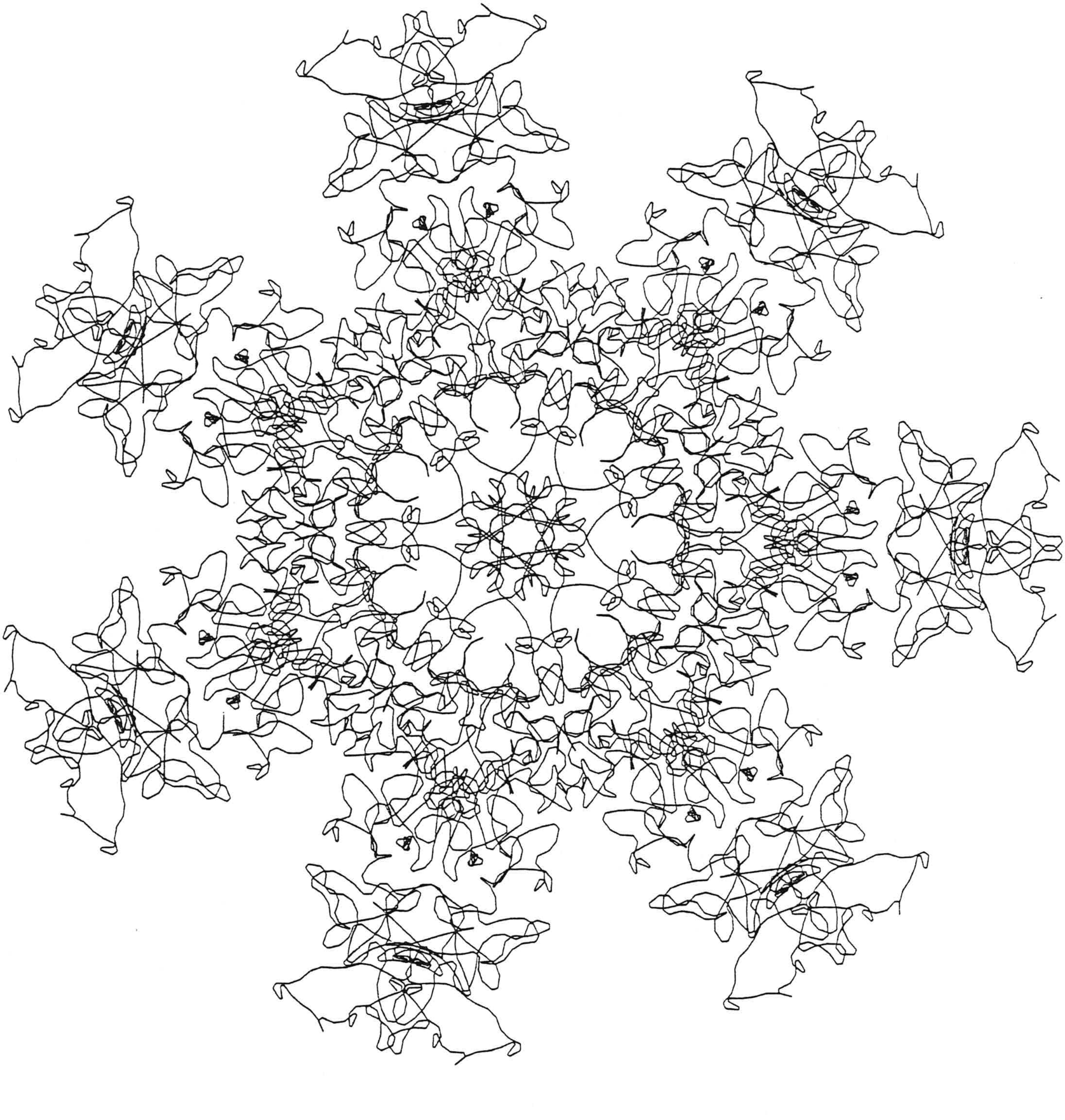
ADDITIONAL ILLUSTRATIONS (concl.)

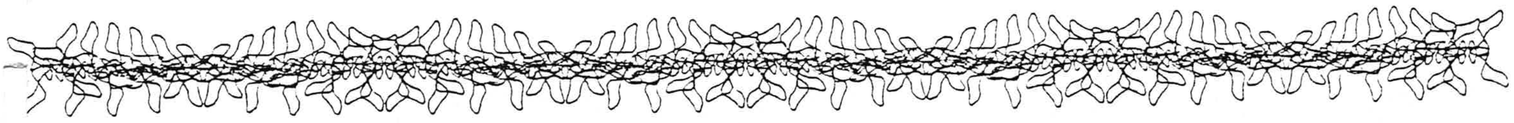
figure	n	s	j_0	α	sym
xxii	$31^3 = 39,791$	8	0	31	L
xxiii	"	10	0	"	L
xxiv	"	12	"	"	"
xxv	"	14	"	"	"
xxvi	"	16	"	"	"
xxvii	"	100	"	"	"
xxviii	"	102	"	"	"
xxix	"	104	"	"	"
xxx	"	108	"	"	"
xxxi	"	106	"	"	"
xxxii	"	110	"	"	"
xxxiii	"	112	"	"	"



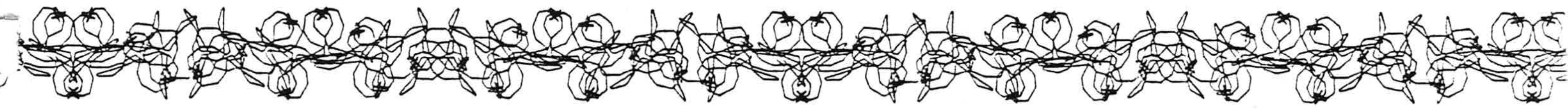








v-a



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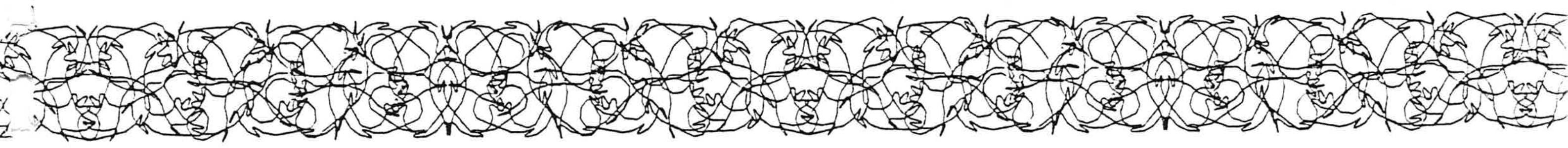

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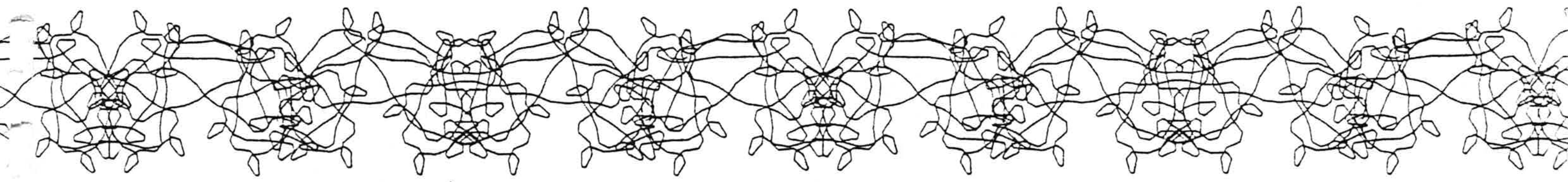

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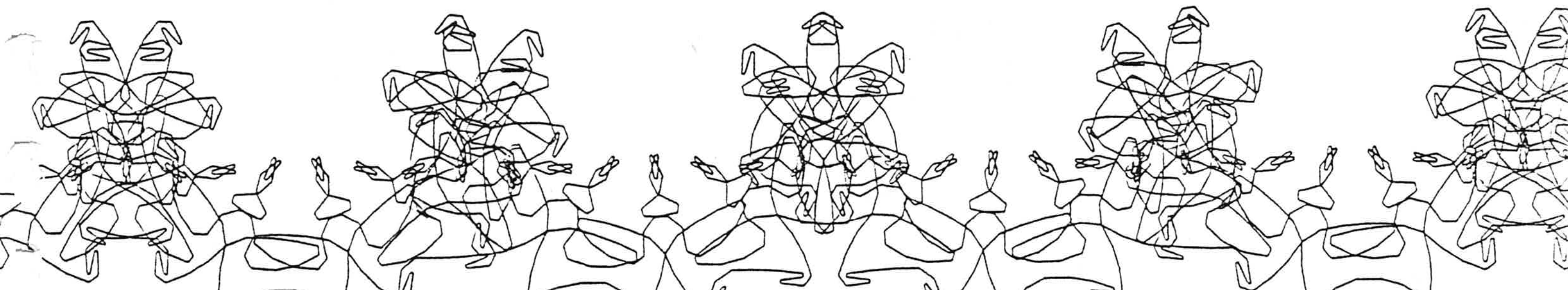

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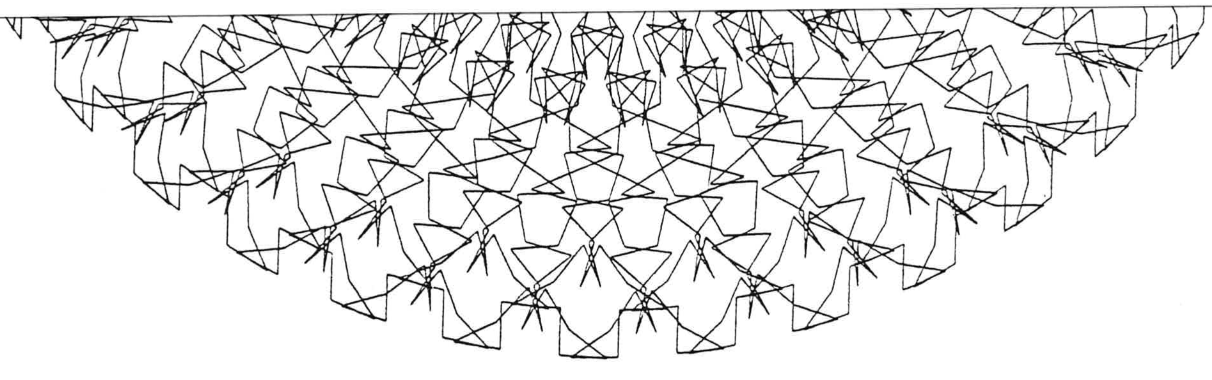

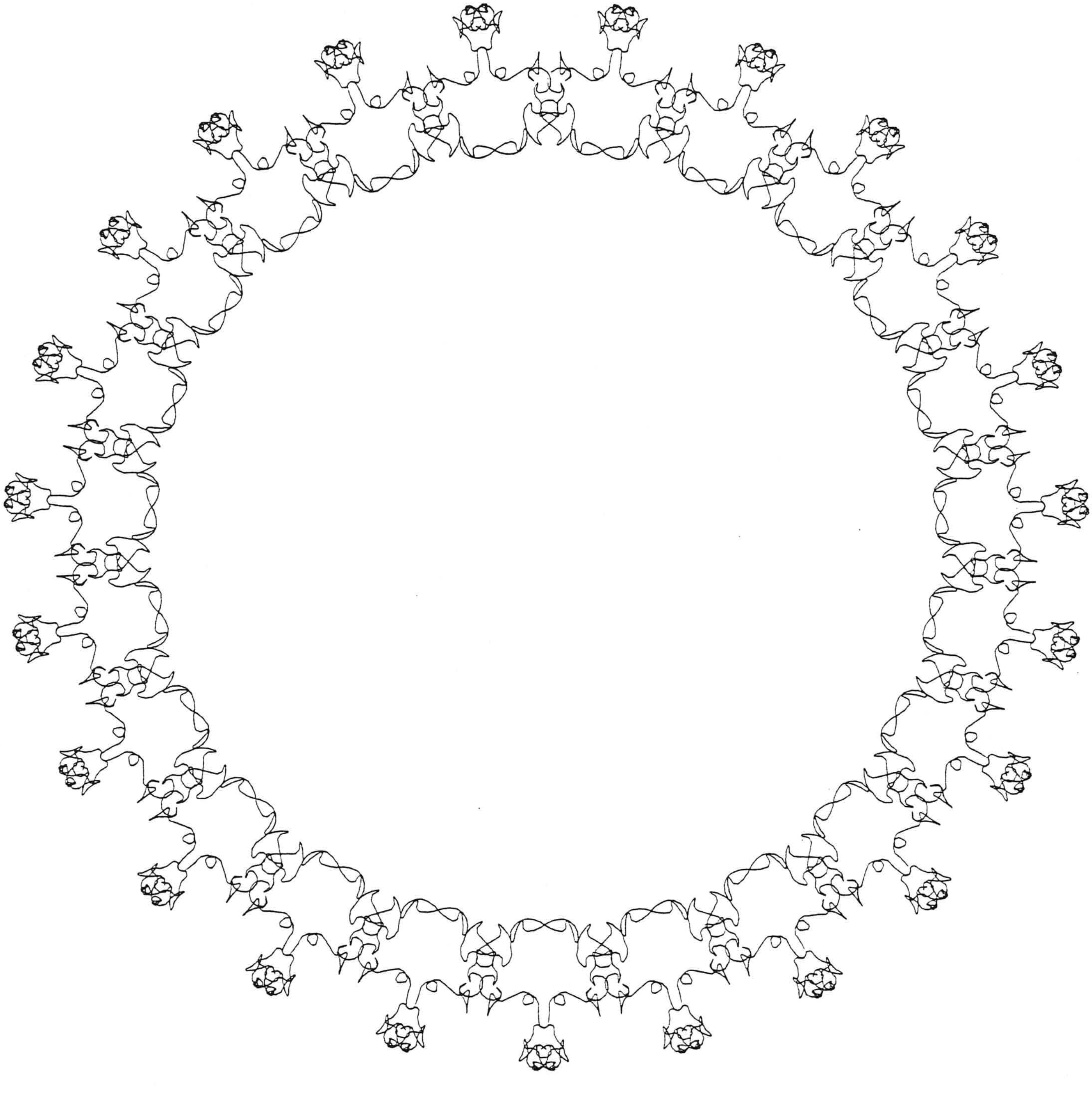

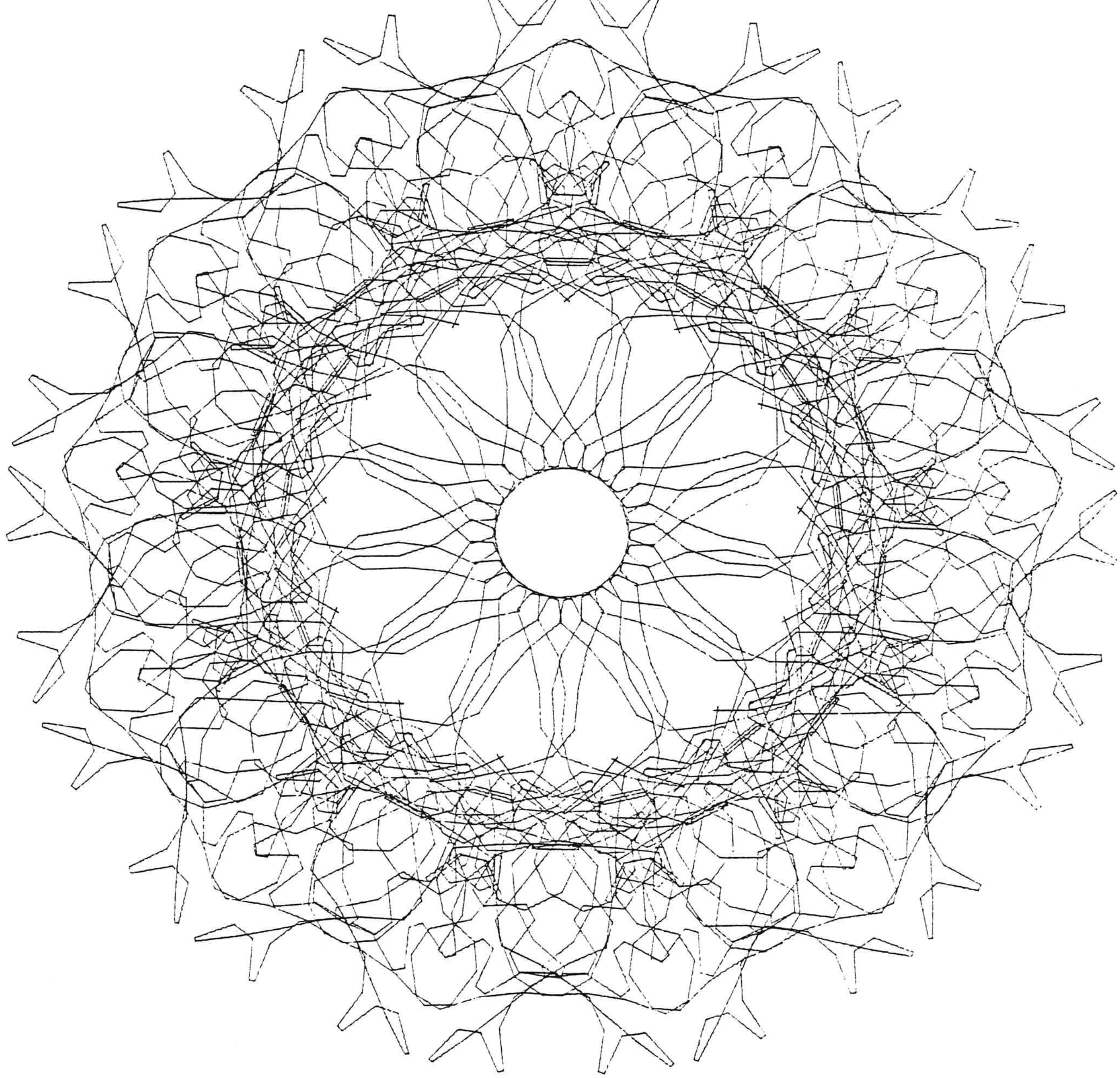

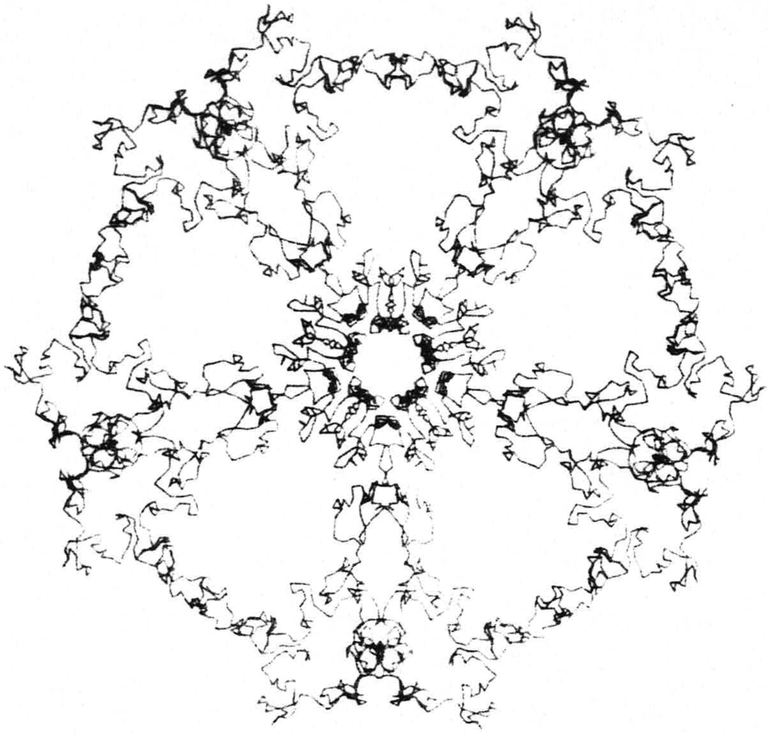

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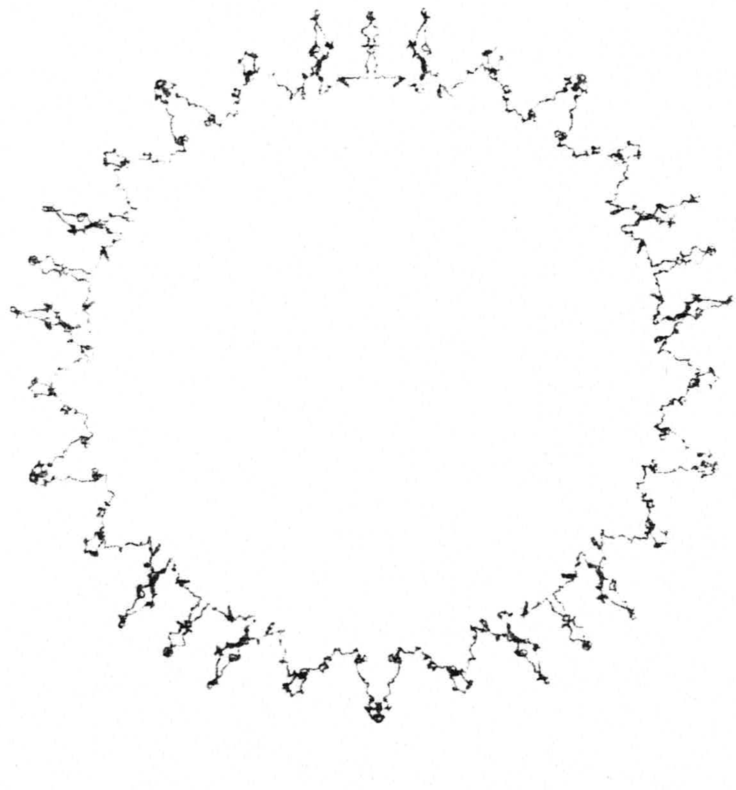

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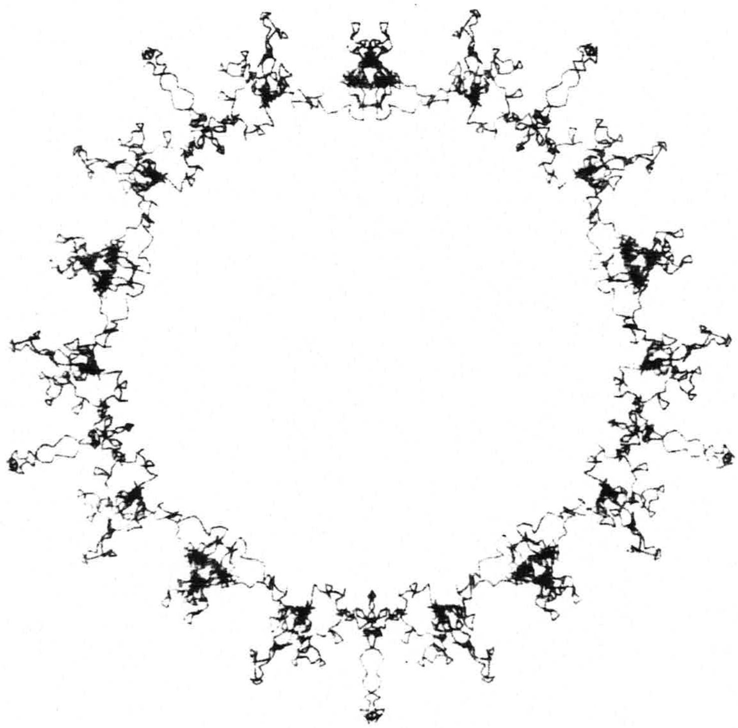

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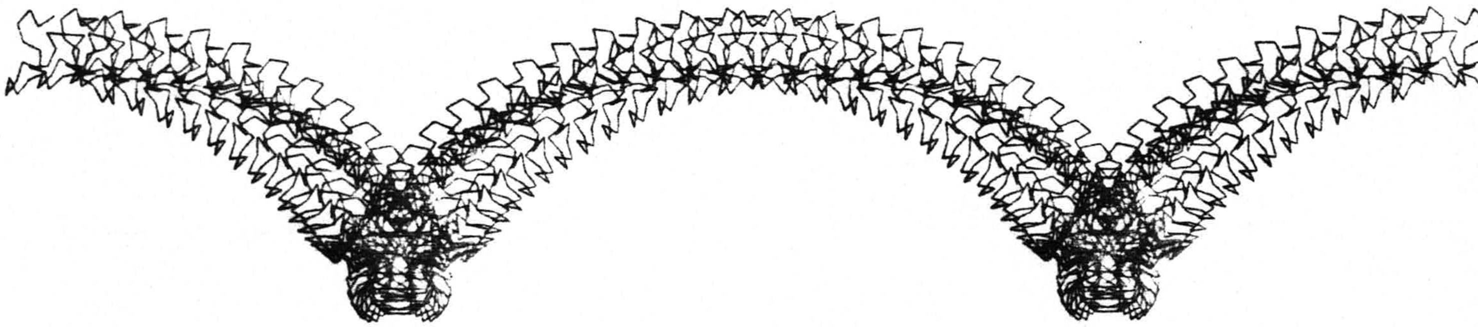

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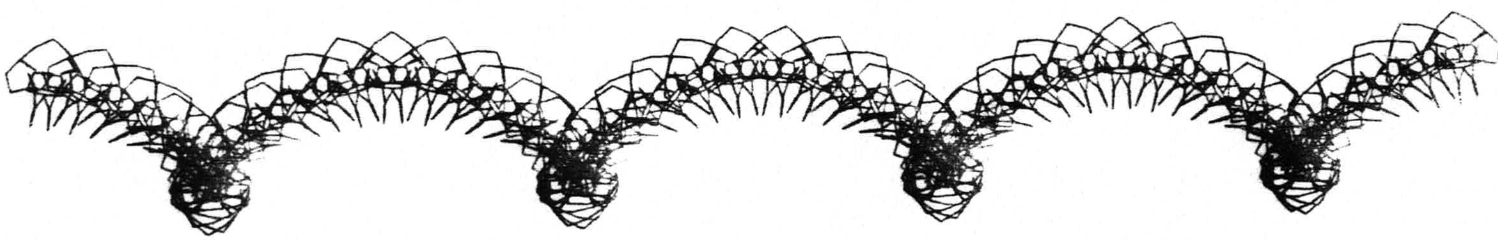

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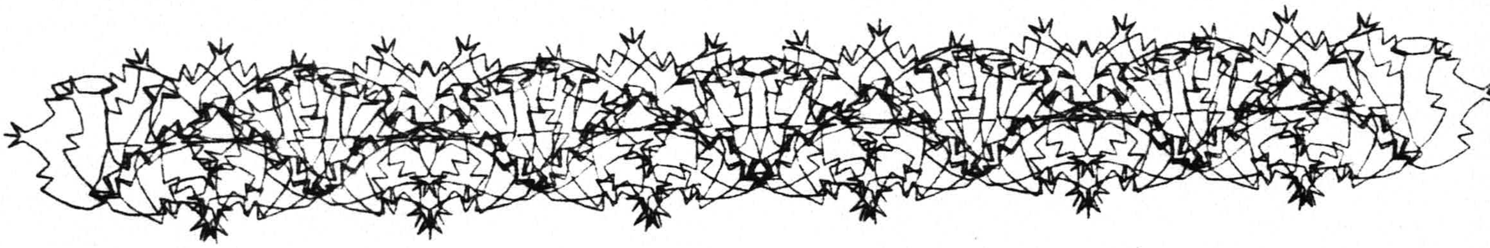

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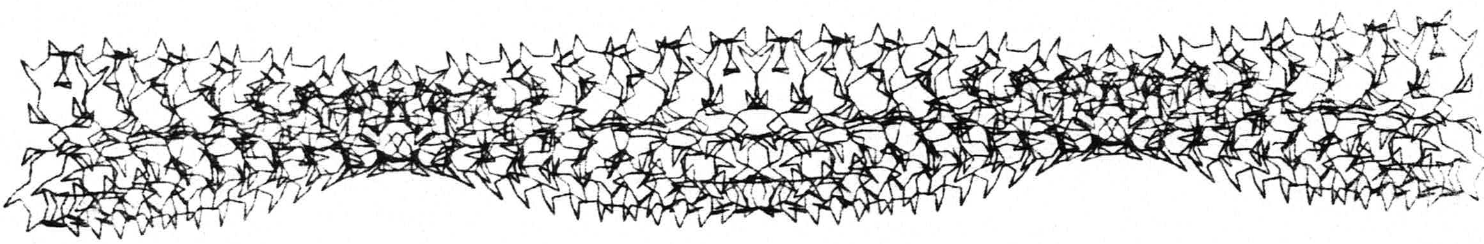

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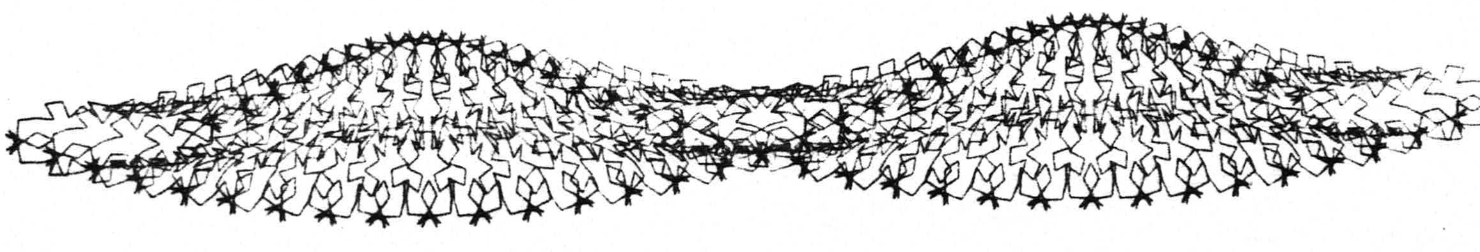

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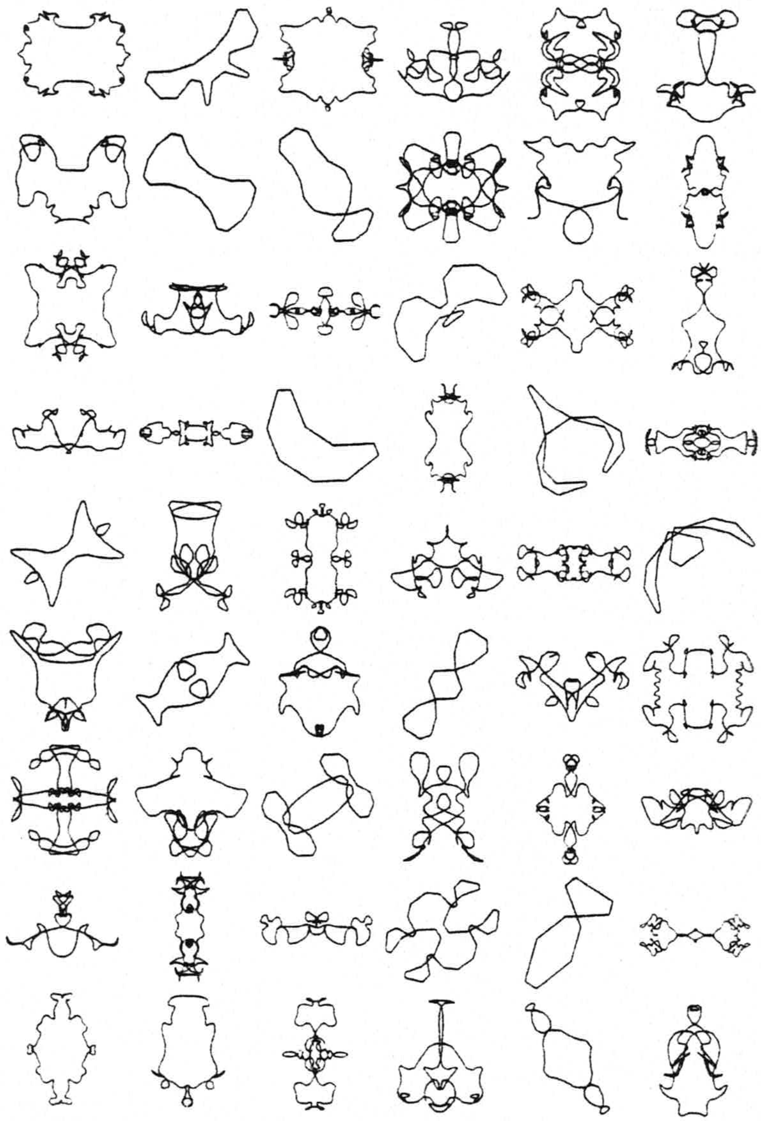

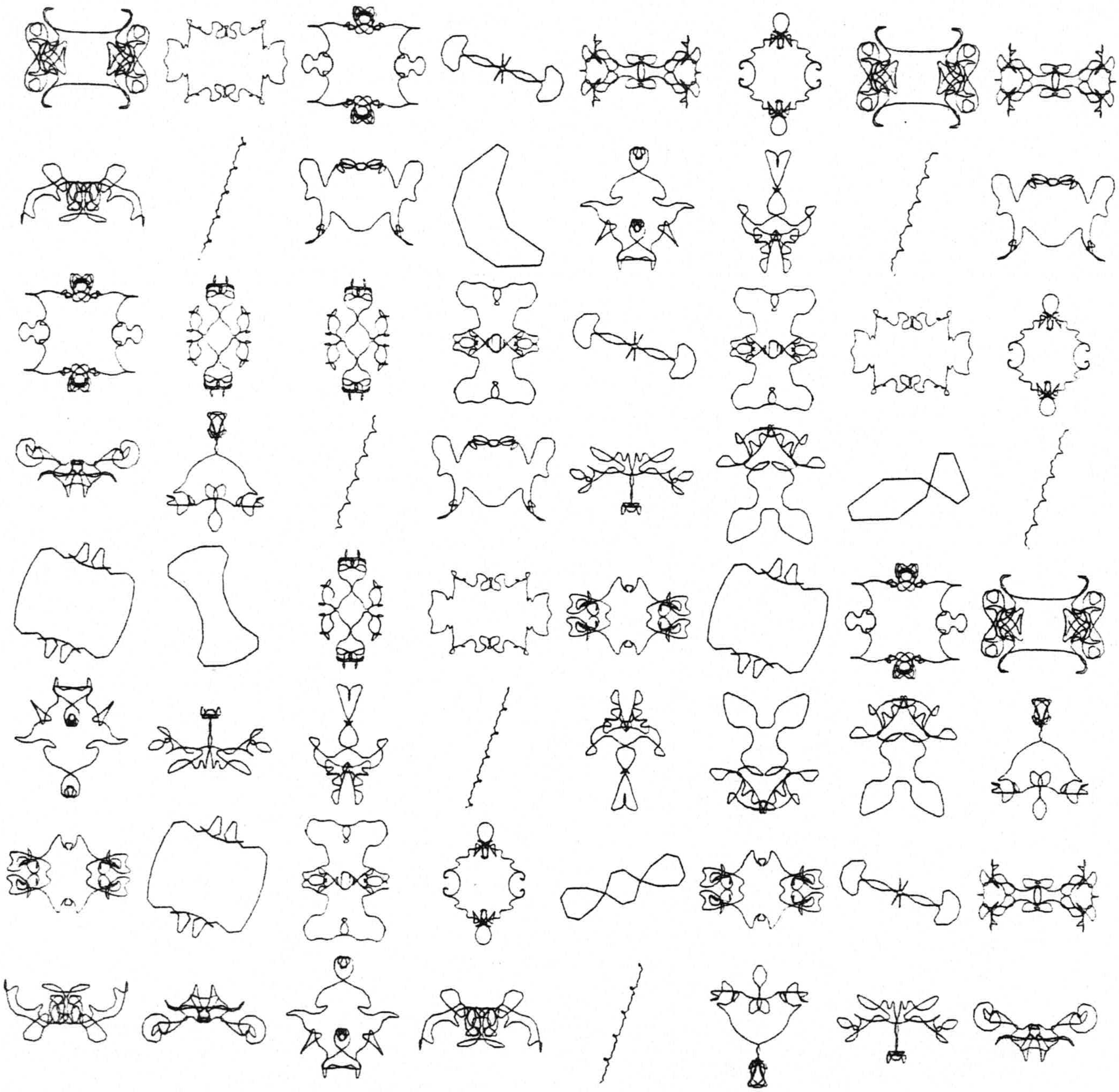

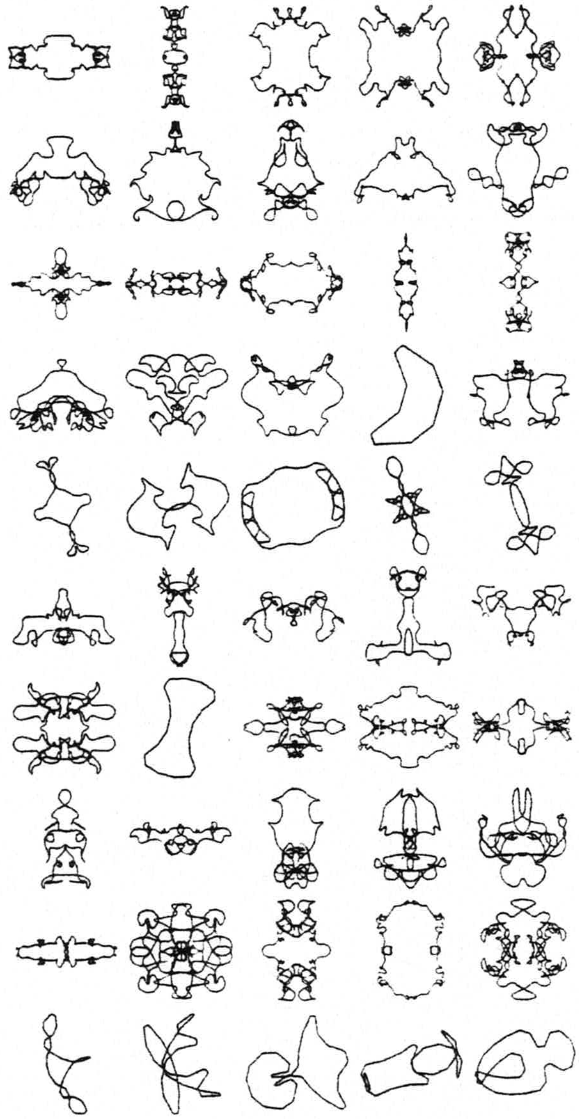

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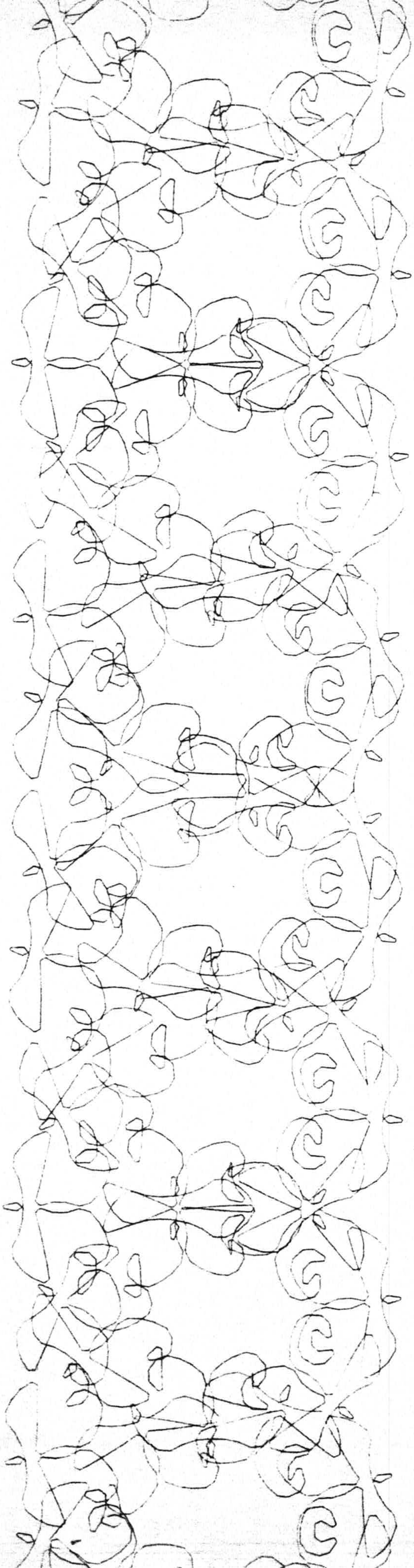

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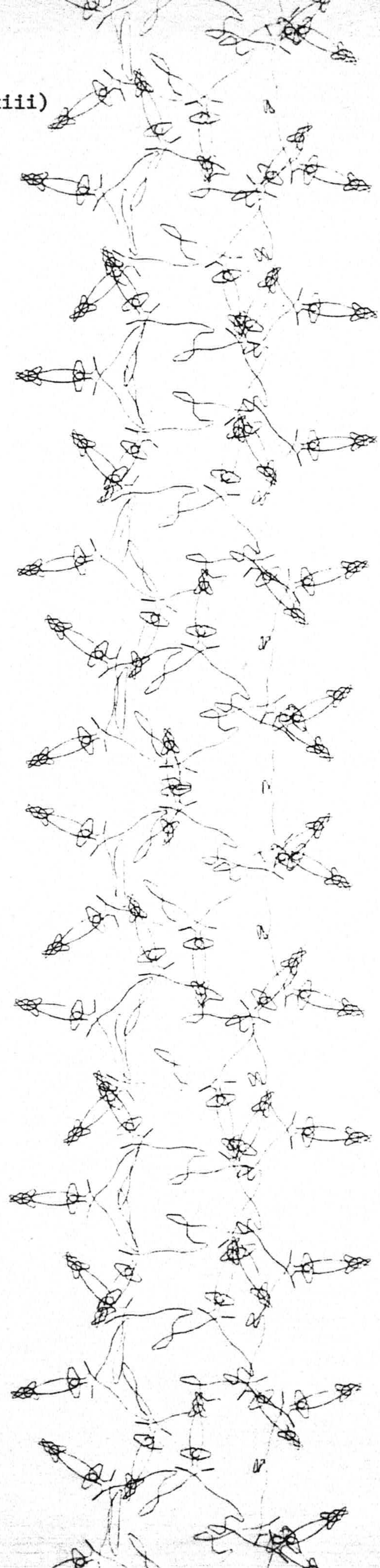

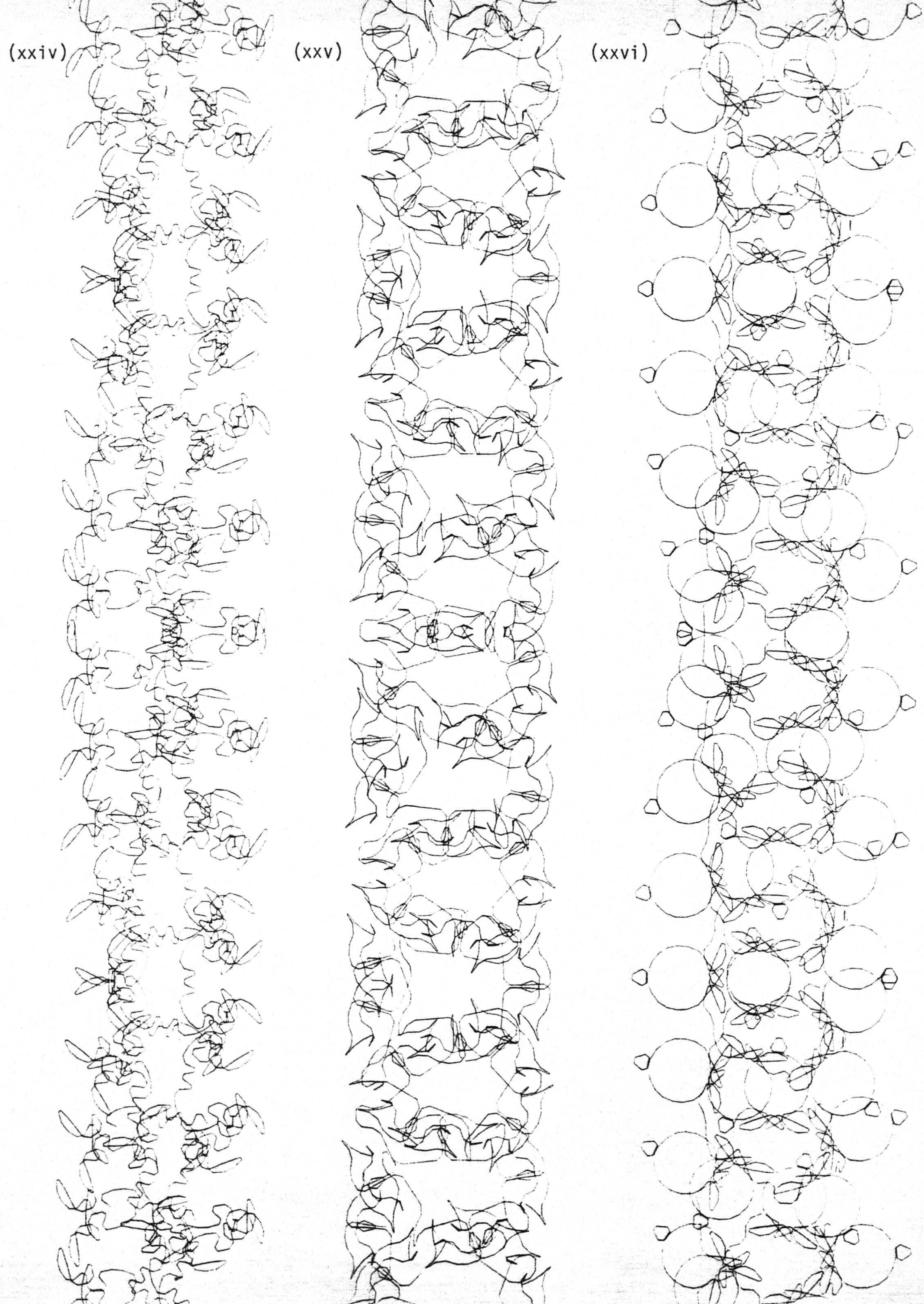
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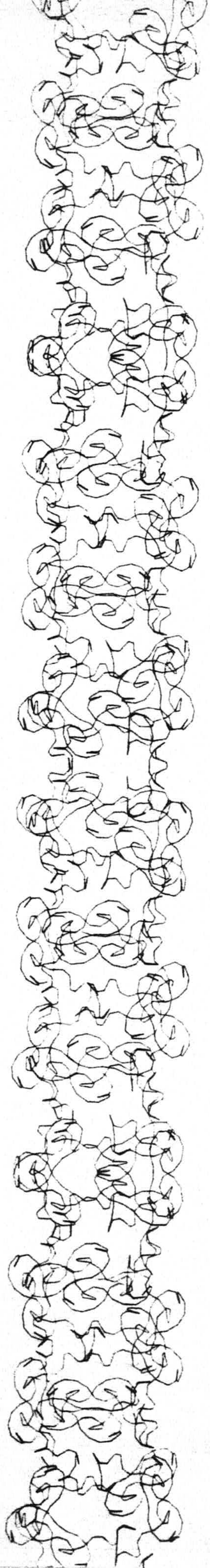

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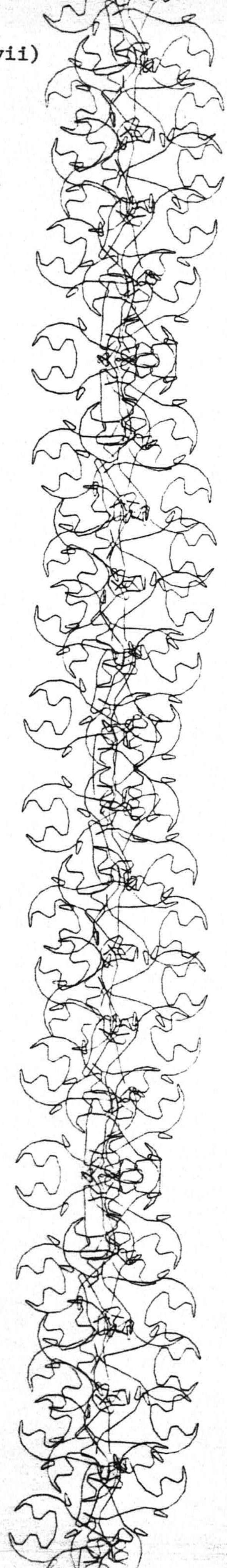

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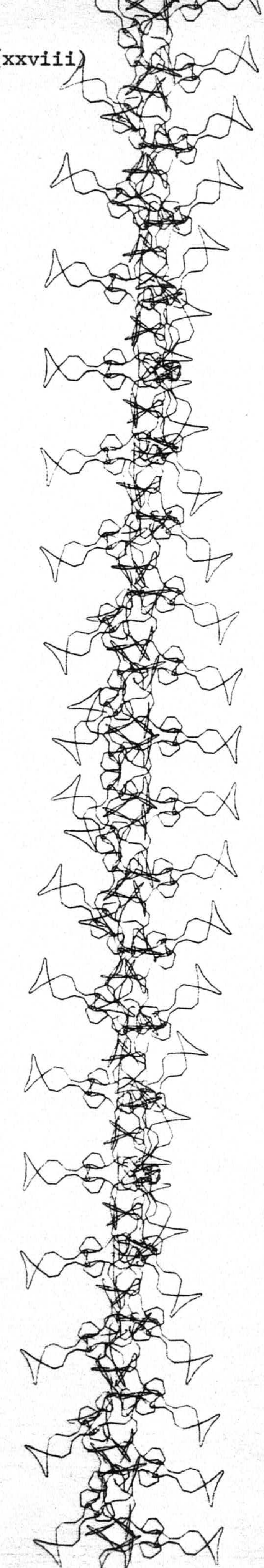

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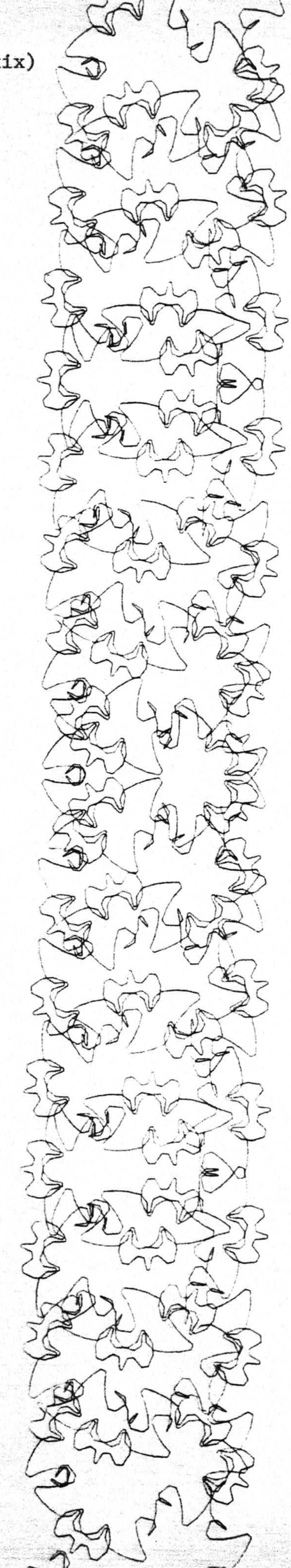

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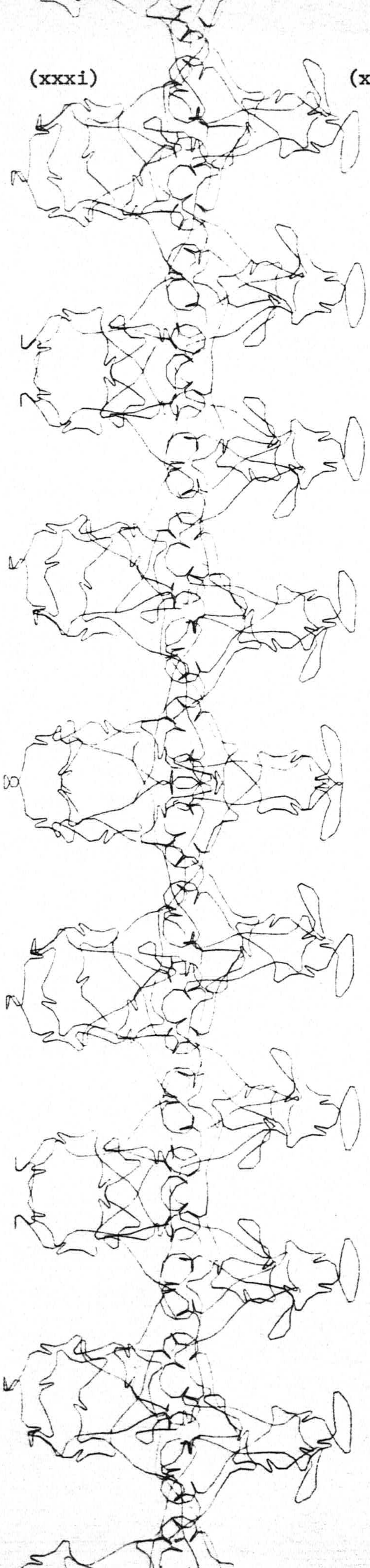

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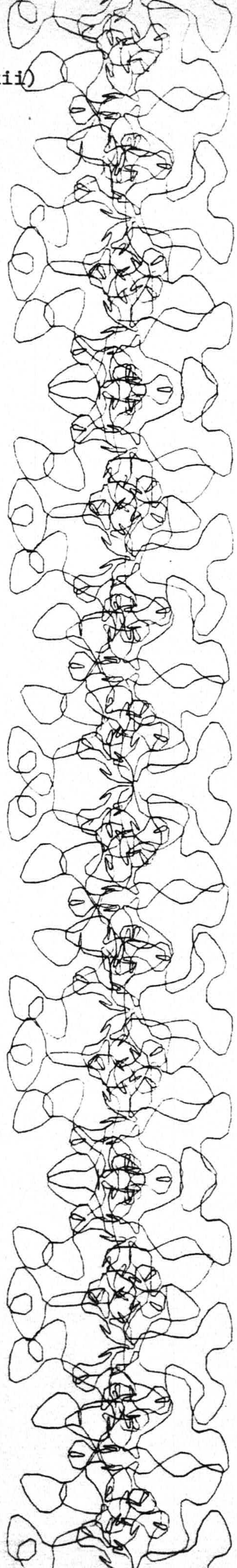

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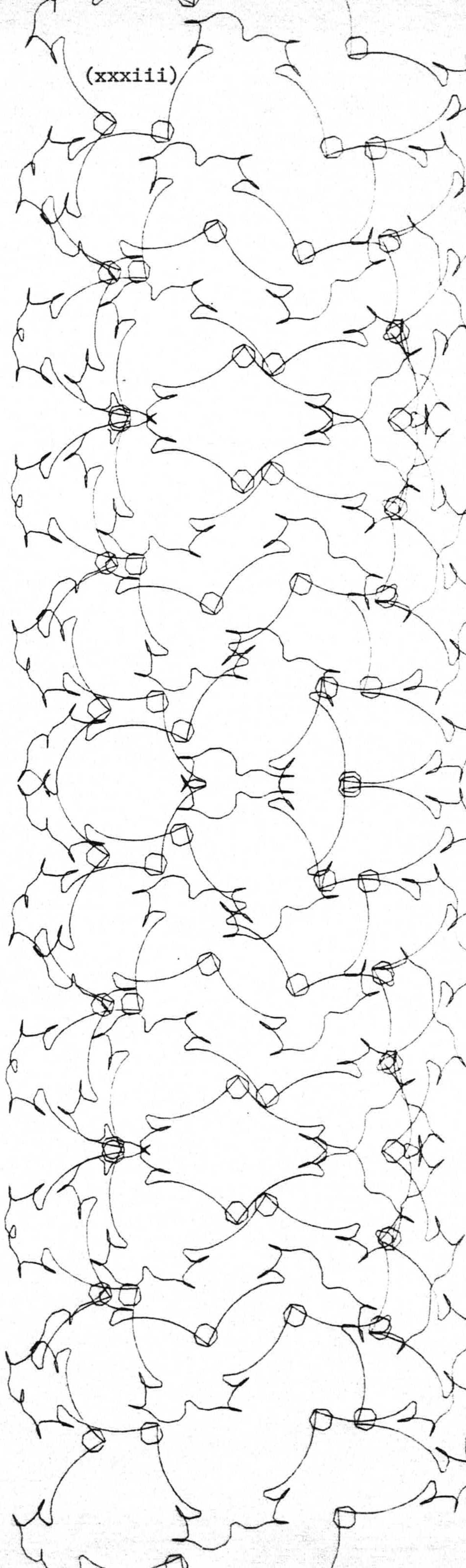

I. Introduction

These remarks provide a brief summary of a study, not yet published, of a special class of ornamental designs which I call K-patterns. These designs can be produced rapidly and with high precision on a personal computer equipped with a graphics monitor. Throughout this note, K-patterns will be defined as the infinite set of 2-dimensional designs constructed by joining the points corresponding to consecutive partial sums of the set $\{r_k\}$, where

$$r_k = \sum_{j=0}^{k} \exp\,[i\,(j_o + sj)^{\alpha}\pi/n]; \qquad \text{(I.1)}$$

$k = 0, 1, 2, \ldots$, fundamental period (hereafter called simply *period*);

the *parameters* α, n, j_o, s are positive integers.

K-patterns are unlimited in their variety. They include both strip designs (1-dimensional lattice symmetry) and centrosymmetric designs (dihedral symmetry of every order and cyclic symmetry of every order). Many of the patterns of low symmetry resemble cartoons of animals or human beings.

K-patterns* may be said to belong to *finite differential geometry*: each portion of a curve is generated by moving a point through a finite distance in a specified direction. It is characteristic of such locally defined curves that it may be difficult to determine their global properties (in the case of K-patterns, their symmetry, period, and--if the curve is closed--star density [winding number]).

The connections between K-patterns and LOGO-generated graphics are apparent. In fact, a number of LOGO "spiral curves" which have appeared as illustrations in articles or books on LOGO were already familiar to me as K-patterns corresponding to small integer values of the four parameters.

It may not be obvious why I have not chosen to simplify the definition of K-patterns by setting $j_o = 0$ in Eq. I.1. Such a choice would simplify the analysis of both the symmetry and period of K-patterns, but at the cost of reduced pattern variety. It would also complicate considerably

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*I chose the name K-patterns because of the investigations by E. Kummer of number-theoretic questions involving power residues, first mentioned to me by Andrew Odlyzko.

the control of color reflection symmetry (discussed in §IV). Specifically, three reasons for allowing non-zero values of j_o are as follows:

(i) For the K-patterns decribed in §III.A of this note, whenever (n, s) > 1, the number of non-trivial examples is severely reduced if $j_o = 0$ instead of $j_o = 1$, for example.

(ii) If α is odd, then no K-patterns with cyclic symmetry are possible, unless $j_o \neq 0$.

(iii) If the symmetry is either lattice or dihedral, flexibility in the choice of j_o makes it a simple matter to generate K-patterns with perfect *color reflection symmetry.*

The way I have chosen to add color reflection symmetry to initially monochromatic K-patterns may be described as follows: If the pattern has d_1 symmetry, then it remains monochromatic. If it has d_k symmetry with $k \geq 2$, then adjacent replicas (*sectors*) of the fundamental region of the pattern's *rotation* group (of order k) are colored differently, within the limitations of the palette of available colors. If the pattern has lattice symmetry L (group generated by two parallel reflections), then adjacent replicas (cells) of the fundamental region of the pattern's translation group are colored differently. In any case, the color boundaries of each replica of the fundamental region of the pattern coincide with lines of reflection of the corresponding monochromatic pattern.

For each of a number of infinite *families* of patterns, I have devised algorithms (*p-s rules*) for calculating the period and symmetry of any K-pattern in the family from the parameter values. These rules were developed both from empirical computer runs and by the application of elementary number theory. The reason for developing such rules is a practical one: without advance knowledge of both period and symmetry, it would be impossible to reduce the generation of a properly centered, scaled, and colored plot to a fast automatic process suitable for a personal computer.

In the case of centrosymmetric patterns, if both the period and the order g of the symmetry group are known before the positions of pattern points are computed, it then becomes possible to reduce the time required for the pattern centering and scaling computations which must precede plotting. For $g \gg 1$, this saving can be appreciable: the required time is reduced by a factor which approaches g.

II. Six theorems and two conjectures

A. Theorems

I have proved a variety of theorems about K-patterns. Six of the ones with relatively easy proofs are as follows:

Theorem 1. Every K-pattern is periodic in the summation index j, with period 2n. The *fundamental* period may be less than 2n.

Theorem 2. Let α = any positive integer ≥ 2. Factor n and s as follows:

$$n = \prod_{i=1}^{r} Pi^{\nu_i} \quad ; \qquad \text{(II.1)}$$

$$s = s_0 \prod_{i=1}^{r} Pi^{\sigma_i} \quad ; \qquad \text{(II.2)}$$

$$s_0 = s \,/\, [n, s]. \qquad \text{(II.3)}$$

Then the smallest integer m which satisfies

$$[s\,(k + m)\,]^{\alpha} \equiv (sk)^{\alpha} \pmod{2n} \qquad \text{(II.4)}$$

for all integer values of k, n, and s is given by

$$m = \prod_{i=1}^{r} Pi^{\mu_i} \quad ; \qquad \text{(II.5)}$$

where

$$\mu_i = \max\{\, 0 \cup (\nu_i - \alpha\sigma_i)\,\} \; ; \qquad \text{(II.6)}$$

and

$$\epsilon = 2 \text{ if n and s are both odd;} \qquad \text{(II.7)}$$

ϵ = 1 otherwise.

Theorem 3. For any integer values of α and of n, the K-pattern for j_0 = 1, s = 1 has period 2n. It is symmetrical under inversion in thecenter of the pattern. If n and α are both odd, the pattern is a closed curve which is symmetrical by reflection in two perpendicular lines through its center.

Theorem 4. If n is prime, s < n, and s' = 2n - s, then for any α and any j_0, the K-patterns for s and s' are related by reflection in the x-axis.

Theorem 5. If n is prime, then for any α the K-patterns for $j_o = 0$ and $j_o = 1$ are identical, except that

(i) for *odd* s, there is a phase shift h between the indexing schemes for the two patterns:

$$(1 + sj)^\alpha \equiv s^\alpha (j + h)^\alpha \pmod{2n}; \qquad \text{(II.8)}$$

h satisfies the linear congruence

$$sh \equiv 1 \pmod{2n}; \qquad \text{(II.9)}$$

(ii) for *even* s, each pattern is rotated through a half-turn with respect to the other, and in addition there is a phase shift λ between the indexing schemes for the two patterns:

$$(1 + sj)^\alpha \equiv [s^\alpha (j + \lambda)^\alpha + n] \pmod{2n}; \qquad \text{(II.10)}$$

λ satisfies the linear congruence

$$s\lambda \equiv 1 \pmod{n}. \qquad \text{(II.11)}$$

Theorem 6. Consider any K-pattern for $\alpha = 3$ whose symmetry group includes one or more reflection isometries.

Let m = period;

sym = order of symmetry group if symmetry is dihedral;

sym = 1 if symmetry is lattice (group generated by two parallel reflections).

Define the *complexity* of any K-pattern as

$$\text{complexity} = m/\text{sym}. \qquad \text{(II.12)}$$

Then perfect color reflection symmetry results if either

(a) n and s are both even or both odd,

$$3s^2 \cdot (n + s \cdot \text{complexity}) \equiv 0 \pmod{2n}, \qquad \text{(II.13)}$$

and

$j_o = (n + s)/2$; or

(b) s is even,

$$3s^3 \cdot \text{complexity} \equiv 0 \pmod{2n}, \qquad \text{(II.14)}$$

and

$j_o = s/2$.

cyclic K-patterns based on cubic residues.

B. Conjectures

In the area of polynomial residues, I have formulated two conjectures which predict K-pattern symmetry and period according to the parity (even or odd) of the sum of the (integer) coefficients of the polynomial. The results of many tests support these conjectures without exception. (I have not attempted proofs.)

Conjecture 1.

Let n = odd prime,

and $Q(j) = \sum_{i=1}^{t} c_i j^{\alpha_i}$;

α_i = odd integer, and c_i = positive integer or zero (i = 1, 2, . . ., t).

Let $C = \sum_{i=1}^{t} c_i$.

Now consider the K-patterns for *polynomial residues,* in which r_k of Eq. (I.1) is replaced by

$$r_k = \sum_{j=0}^{k} \exp\,[i\,Q(j)]\,\pi/n.$$

Then

if s is even, or s is odd and C is even,

symmetry = L, and period = n;

if s is odd and C is odd,

symmetry = d_2, and period = 2n.

We define symmetry = L to mean that the symmetry is that of a lattice generated by two parallel lines of reflection (perpendicular to the translation axis).

Conjecture 2.

Let n = odd prime,

and $Q(j) = \sum_{i=1}^{t} c_i j^{\alpha_i}$;

α_i = even integer, and c_i = positive integer or zero (i = 1, 2, . . . , t).

Let $C = \sum_{i=1}^{t} c_i$.

Now again consider the K-patterns for polynomial residues, in which r_k of Eq. (I.1) is replaced by

$$r_k = \sum_{j=0}^{k} \exp [i\, Q\, (j)] \pi / n.$$

Then

if s is even, or s is odd and C is even,

symmetry = L*, and period = n;

if s is odd and C is odd,

symmetry = C_2, and period = 2n.

We define symmetry = L* to mean that the symmetry is that of a lattice generated by two half-turns (their centers lying on the translation axis).

III. Examples of pattern families

Among the several infinite families of K-patterns for which I have developed complete p-s rules are the following:

A. $j_0 = 1$; $1 < \alpha < 7$; n, s are any positive integers.

For each of these seven sub-families, the p-s rules are somewhat too complicated for a brief summary. In any event, both the execution time and precision for programs which implement them are acceptable even on a personal computer operating in interpretive BASIC, without ASSEMBLER subroutines to speed things up. There is, however, one qualification. Consider the case of cubic residues ($\alpha = 3$). When the p-s rules, as originally developed, predicted that a given $\langle n, s \rangle$ integer pair corresponded to a pattern with lattice symmetry, it was sometimes found that the symmetry was actually d_1. The algorithm was then modified to accommodate such exceptions until no more could be found. Subsequent testing has uncovered no further errors. For the cases $\alpha = 5$ and $\alpha = 7$, I have not yet tested all of the analogous algorithm corrections. For $\alpha = 1$, the problem does not even arise. I have not yet examined this aspect of the cases $\alpha = 2, 4$, and 6, but I intend to do so when time permits.

B. α = odd integer > 1

1. n = prime $\ni (n, \alpha) = 1$, and $n \not\equiv 1 \pmod{\alpha}$.

a. s = 1, 3, 5, . . . , n - 2

period = 2n; (III.1)

symmetry = d_2.

b. s = 2, 4, 6, . . . , n - 1

period = n; (III.2)

symmetry = d_1.

For both even and odd s, no two patterns are alike; hence there are n - 1 distinct patterns.

2. <u>n = prime $\ni$ $(n, \alpha) = 1$, and $n \equiv 1 \pmod{\alpha}$.</u>

a. <u>$s = 1, 3, 5, \ldots, n-2$</u>

$$\text{period} = 2n; \qquad \text{(III.3)}$$

symmetry = d_2.

b. <u>$s = 2, 4, 6, \ldots, n-1$</u>

$$\text{period} = n; \qquad \text{(III.4)}$$

symmetry = lattice (L).

For both even and odd s, each pattern occurs for α different values of s; hence there are only $(n - 1)/\alpha$ distinct patterns.

Let t_1 (= 1), $t_2, \ldots, t_\alpha$ = the α different *integer* α^{th} *roots of unity* (mod $2n$), i.e., the integer solutions of the cyclotomic equation (mod $2n$)

$$s^{\alpha} \equiv 1 \pmod{2n} \qquad \text{(III.5)}$$

Then if a given pattern occurs for $s = s_1$, it also occurs for

$$s_i \equiv t_i s_1 \pmod{2n} \quad (i = 2, 3, \ldots, \alpha). \qquad \text{(III.6)}$$

C. n = square-free odd integer

$= p_1 p_2 \cdots p_r$;

$\alpha = p_w$ $(w = 1, 2, \ldots, r)$;

$j_0 = 0$;

$s = 1, 2, \ldots, n - 1$.

Then

$$\text{period} = \epsilon n/(n, s), \qquad \text{(III.7)}$$

where

$\epsilon = 1$ if s is even,

and (III.8)

$\epsilon = 2$ if s is odd;

$$\text{symmetry} = d_q, \qquad \text{(III. 9)}$$

where

$$q = \epsilon \prod_{i=1}^{r}{}' \; [p_i/(p_i, s)]; \qquad \text{(III.10)}$$

the product $\prod'$ includes only those p_i for which $(p_i - 1) | (p_w - 1)$.

Eq. (III.10) was constructed after perusing the article by A.E. Livingston and M.L. Livingston: "The congruence $a^{r+s} \equiv a^r \pmod{m}$" (Amer. Math. Monthly 85 (1978), pp. 97-100.

I have not proved the results expressed in Eqs. (III.7-10). I have merely verified them for a respectable number and variety of examples.

D. $n = p^3$ (p = odd prime);

$\alpha = p$;

$j_o = 0$.

1. $s = 1, 3, 5, \ldots, p^2$

period $= 2p^2$; symmetry $= d_1$. (III.11)

Pattern consists of two distinct images of a design D with d_p symmetry, plus horizontal unit vectors (connecting lines) which join these two images. Each of the rotational fundamental regions (pattern "motifs") of D is made up of p-1 unit vectors; each image of this motif is rotated through the angle $2\,(sp^2 \pmod{2p^3})\,\pi/p^3$ with respect to the previous image. Every p^{th} unit vector of the pattern is a connecting line; it is directed alternately to the left or right.

2. $s = 2, 4, 6, \ldots, p^2 - 1$

period $= p^2$; symmetry = lattice (L). (III.12)

A single translational cell of the pattern consists of p replicas of a motif comprising p-1 unit vectors, plus p left-directed unit vectors (connecting lines). Each image of this motif is rotated through the angle $(sp^2 \pmod{2p^3})\,\pi/p^3$ with respect to the previous image. Every p^{th} unit vector (connecting line) is directed to the left, thereby joining each image of the pattern motif to the next. The skeleton of each pattern is a prolate or curtate cycloid, depending on the magnitude of the resultant of the p-1 unit of the resultant of the p-1 unit vectors of the pattern motif.*

For this family of patterns, I have proved all of the described properties.

E. $n = q^2$ (q = odd integer $\ni 3 \nmid q$);

$\alpha = 3$;

$s = q/2^{\ell}$ ($\ell = 1, 2, \ldots$);

$j_o = k/2^{\ell}$ ($k = 1, 2, \ldots, (q-1)/2$);

period = $16\, q\, \ell^3$;

symmetry = d_q.

For $\ell = 1$, the density of the skeletal star polygon (winding number) = $3k^2 \pmod q$.

For each ℓ, there are $(q-1)/2$ distinct patterns, corresponding to the $(q-1)/2$ allowed values for j_o.

For this family of patterns, I have proved all of the described properties.

F. $n = qp^2$ (p, q = odd primes $\neq 3$; $(p,q) = 1$; $\alpha = 3$)

(i) $s = 2kp$ $(1 \leq k \leq p - 1)$

period = qp; symmetry = d_p; winding number = $3k \pmod p$.

(ii) $s = kp$ $(o \leq k \leq p - 1)$

period = $2qp$; symmetry = d_{2p}; winding number = $3k \pmod{2p}$.

G. $n = qp^3$ (p, q = odd primes $\neq 3$; $(p, q) = 1$; $\alpha = 3$)

(i) $s = 2kp$ $(1 \leq k \leq p - 1)$

period = qp^2; symmetry = c_p; winding number = $3k \pmod p$.

(ii) $s = kp$ $(o \leq k \leq p - 1)$

period = $2qp^2$; symmetry = c_{2p}; winding number = $3k \pmod{2p}$.

---

*In some of these examples, the patterns are reminiscent of transformations seen in the graphic designs of M. Escher. Others are suggestive of M. Duchamp (e.g., "Nude Descending a Staircase").

IV. The implementation of color reflection symmetry in a K-pattern

The attractiveness of a K-pattern may be considerably enhanced if it is drawn with two or more colors, especially if the resulting pattern has *color symmetry*. If a monochromatic pattern has *cyclic* symmetry, then all the vectors in any single replica of the pattern motif (*rotational sector**) will be colored alike. To the extent permitted by the number of colors in the palette, we will require strict color symmetry: each rotational isometry of the monochromatic pattern must permute the sector colors according to a self-consistent scheme.

Next consider the problem of expressing color reflection symmetry in K-patterns whose symmetry group includes one or more reflection isometries. The solution of this problem is effected by choosing a suitable integer value for j_0 to replace the value 1 on which the p-s rules described in III.A (p. 8) were based. Of course it is necessary to insure that the pattern resulting from this new value of j_0 is the same -- aside from color -- as the one generated with $j_0 = 1$. The simplest way to guarantee this equivalence is to restrict j_0 to the set $\{1 + Zs\}$ $(Z = 0, 1, 2, \ldots)$. Now let us specify coloring procedures for the various cases.

1. For K-patterns with d_1 symmetry, we will use one color only.

2. For K-patterns whose symmetry is described by the lattice group L, generated by two parallel reflections, we will simplify this discussion (and also the treatment of *dihedral* patterns which follows) by defining K-patterns with either d_k symmetry $(k > 1)$ or lattice (L) symmetry *to have property Q* iff

   a) the boundaries of each *rotational sector* or *unit cell*** coincide with lines of reflection;

and

   b) every pattern vector is assigned a single color.

---

*fundamental region of the rotation group of a centrosymmetric pattern.

**fundamental region of the translation group of a lattice pattern.

If the complexity* (cf. Theorem 6, p.5) of a lattice pattern is *odd*, and we require the pattern to have property Q, then if the pattern is to express polychromatic color reflection symmetry, the boundaries of the unit cell can be chosen in only one way. The proof is trivial. (Cf. Fig. f on p. 16c.)

If the complexity of a lattice pattern is *even*, then we might imagine *a priori* that there are two possible cases:

(a) the unit cell includes no pattern vectors (let us call them type $\vec{v}'$) which are parallel to the translation axis, and which are bisected by a line of reflection;

(b) the unit cell includes one pair of pattern vectors of type $\vec{v}'$, and each member of this pair is bisected by a different one of the two translationally inequivalent generating lines of reflection.

In either of the cases (a) or (b) just cited, there would be two complementary choices for the boundaries of a unit cell - either two replicas (related by a unit translational isometry) of one of the two generating lines of reflection, or else two such replicas of the other generating line of reflection.

I have found that among cubic K-patterns, there are no lattice patterns of type L with even complexity which satisfy condition (a) (described immediately above).** Accordingly, L patterns with even complexity cannot have property Q. In order for such a pattern to express polychromatic color reflection symmetry, it is sufficient to divide each vector of type $\vec{v}'$ into two collinear *half* - unit vectors, assigning to each half the color shared by the unit vectors which belong to the same unit cell. (Cf. Fig. d on p. 16b.)

---

*For both d_1 and L patterns, we define sym = 1.

**I have not proved this assertion, but I have proved (cf. Appendix A) that among K-patterns with $\langle n, s\rangle = \langle 2^{\nu}, \text{even integer}\rangle$ (ν = 1, 2, 3,...), none satisfy condition (a). I have also discovered, but not proved, that for this class of patterns, (i) the complexity is even; and (ii) the symmetry is lattice (L) if $\nu \not\equiv 0 \pmod 3$, and d_1 otherwise.

3. For K-patterns with symmetry d_k $(k \geq 2)$, we will color each replica of the pattern motif (rotational sector) with a single color, beginning and ending on unit vectors which are related by reflection in the mirror line which bisects the sector.

I have found that when k is *even*, it is always possible for a d_k pattern which expresses color reflection symmetry to exhibit property Q. If the complexity is even, there are two ways to accomplish this, corresponding to the two complementary choices for the boundaries of a rotational sector - either two replicas (related by a unit rotational isometry R_1 of the group of rotations of the pattern) of one of the two generating lines of reflection, or else two such replicas of the other generating line of reflection. (Cf. Figs. a and b on p. 16a.)

When k is *odd*, on the other hand, I have found (but not proved) that the situation for d_k patterns is quite analogous to that of the lattice patterns described above: (i) If k is *odd* and the complexity is *odd*, there is always exactly *one* way for the pattern to express polychromatic color reflection symmetry and at the same time exhibit property Q (cf. Fig. e on p. 16c);

(ii) If k is *odd* and the complexity is *even*, property Q is impossible; each rotational sector includes one pair of pattern vectors (we will call them type $\vec{v}''$) which are perpendicularly bisected by lines of reflection, and each member of this pair is bisected by a different one of the rotationally inequivalent lines of reflection. There are two complementary choices for the boundaries of a rotational sector - either two replicas (related by a rotational isometry R_1 of the group of rotations of the pattern) of one of the two generating lines of reflection, or else two such replicas of the other generating line of reflection. (Cf. Fig. c on p. 16b.)

The following table provides a summary of the sym-complexity parity classes for solutions to the color reflection symmetry problem for cubic K-patterns. Coloring is governed by the rules described in the immediately preceding paragraphs. In the third column, C = complexity.

	SYMMETRY	SYM	C	PROPERTY Q?	NO. OF SOLUTIONS
a.	d_k (k even)	even	even	y	2: $j_0 = j_0^{**}$, $j_0^{**} + (C/2)\cdot s$
b.	d_k (k even)	even	odd	y	1: $j_0 = j_0^{**}$
c.	d_k (k odd, 3)	odd	even	n	2: $j_0 = j_0^{*}$, $j_0^{*} + (C/2)\cdot s$
d.	lattice (L)	1	even	n	2: $j_0 = j_0^{*}$, $j_0^{*} + (C/2)\cdot s$
e.	d_k (k odd)	odd	odd	y	1: $j_0 = j_0^{**}$
f.	lattice (L)	1	odd	y	1: $j_0 = j_0^{**}$

The solutions j_0^{*} and j_0^{**} are found by solving the congruences IV. 2 and IV. 6 (cf. pp. 17-18), respectively.

Examples of these six parity classes are shown on the next three pages. Simple patterns were chosen for these illustrations, so that the reader can verify some of the remarks made above by coloring the patterns with two, three, or four colors: two or four colors for a, two colors for b, three colors for c, two colors for the two unit cells of d, three colors for e, and two or three colors for the not quite two unit cells of f.

EXAMPLES OF

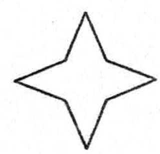
COLOR REFLECTION SYMMETRY

PARITY CLASSES FOR CUBIC K-PATTERNS

Q means that color reflection symmetry can be expressed without assigning different colors to the separate halves of any pattern vector.

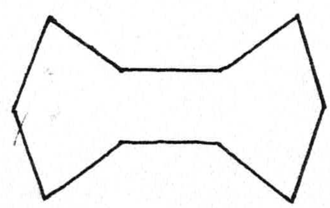
-Q means that color reflection symmetry is expressed by assigning different colors to the separate halves of pattern vectors which are perpendicularly bisected by an alternating subset of the lines of reflection symmetry.

a. $n = 8$
$s = 6$
symmetry = d_4; sym = 4
period = 8
complexity = 2
$j_0^{**} = 1$ or 7
Q



(sym, complexity) = (even, even)

b. $n = 5$
$s = 1$
symmetry = d_2; sym = 2
period = 10
complexity = 5
$j_0^{**} = 3$
Q



(sym, complexity) = (even, odd)

c. $n = 6$

$s = 1$

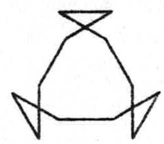
symmetry = d_3; sym = 3

period = 12

complexity = 4

$j_0^* = 2$ or 4

-Q



(sym, complexity) = (odd, even)

d. $n = 16$

$s = 5$

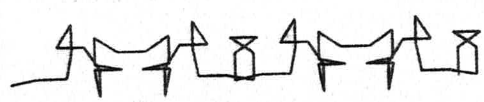
symmetry = lattice (L); sym = 1

period = 32

complexity = 32

$j_0^* = 16$ or 96

-Q



(sym, complexity) = (1, even)

e. $n = 15$

$s = 4$

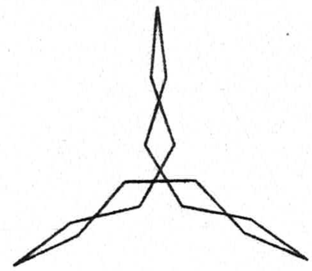
symmetry = d_3; sym = 3

period = 15

complexity = 5

$j_0^{**} = 2$

Q



(sym, complexity) = (odd, odd)

f. $n = 13$

$s = 6$

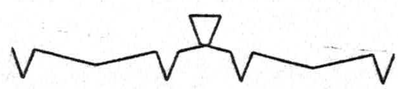
symmetry = lattice (L); sym = 1

period = 13

complexity = 13

$j_0^{**} = 3$

Q



(sym, complexity) = (1, odd)

Now let us address the problem of *implementing* color reflection symmetry in a cubic K-pattern. There are two cases to be considered:

(i) the pattern does not have property Q; then

$\langle$sym, complexity$\rangle$ = $\langle$odd, even$\rangle$.

(ii) the pattern has property Q; then

$\langle$sym, complexity$\rangle$ = $\langle$even, even$\rangle$, $\langle$even, odd$\rangle$, or $\langle$odd, odd$\rangle$.

It is convenient to define sym = 1 for lattice patterns of type L, so that they can properly be included among these four parity classes.

For case (i), let us define j_0^* to be the initial value of the pattern index, i.e., the value of $(j_0^* + sj)$ for j = 0. The pattern vector exp $[i\ j_0^{*}\alpha\pi/n]$ is of type $\vec{v}'$ or $\vec{v}''$, according to whether the symmetry is lattice or dihedral; it is perpendicularly bisected by a line of reflection symmetry. In order for the K-pattern to express polychromatic color reflection symmetry, the rear half of this pattern vector will be assigned the same color (H) as the C-½ unit vectors which immediately precede it, while the forward half will be assigned the same color (H') as the C-½ unit vectors which immediately follow it. (H and H' need not necessarily be different colors, depending on the specific group of color reflection isometries associated with the pattern.) Now let C = complexity. The j_0^* is a solution of

$$[j_0^* + ks]^3 - j_0^{*3} + [j_0^* - ks]^3 - j_0^{*3} \equiv 0 \pmod{2n}; \qquad \text{(IV. 1)}$$

$k = 0,1,2,\ldots,C/2.$

If the l.h.s. of Eq. IV.1 is to vanish identically for $0 \leq k \leq C/2$, it is necessary that

$$6s^2 j_0^* \equiv 0 \pmod{2n}. \qquad \text{(IV. 2)}$$

In order to guarantee that the colored pattern resulting from the choice $j_0 = j_0^*$ be the same (aside from color) as the monochromatic pattern generated by choosing $j_0 = 1$, we require that

$$j_0^* = 1 + Zs \quad (Z = 0, 1, 2, \ldots). \qquad \text{(IV. 3)}$$

Eqs. IV. 2 and IV. 3 yield the result

$$6s^3Z \equiv -6s^2 \pmod{2n}. \qquad \text{(IV. 4)}$$

This linear congruence has a unique solution mod $(2n/(6s^3, 2n))$ if $(6s^3, 2n)\ 6s^2$, and none otherwise.*

For case (ii), let us define j_0^{**} to be the initial value of the pattern index, i.e., the value of $(j_0^{**} + sj)$ for $j = 0$. The pattern vector exp $(i\ j_0^{**}\alpha\pi/n]$ is the first vector of a new rotational sector or unit cell. In order for the K-pattern to express polychromatic color reflection symmetry, the next C-1 pattern vectors are assigned the same color H as this vector. The preceding C pattern vectors are also assigned a single color H'. (H and H' need not necessarily be different colors, depending on the specific group of color reflection isometries associated with the pattern.) Then j_0^{**} is a solution of

$$[j_0^{**} + (k+1)s]^3 - [j_0^{**} + ks]^3 + [j_0^{**} + (C-1)s - (k+1)s]^3 - [j_0^{**} + (C-1)s - ks]^3$$
$$\equiv 0 \pmod{2n}; \qquad \text{(IV. 5)}$$

$k = 0, 1, 2, \ldots, k_{max}$;

$k_{max} = (C-3)/2$ if C is odd;

$k_{max} = (C-4)/2$ if C is even.

In order that the l.h.s. of IV. 5 vanish for $0 \leq k \leq k_{max}$, it is necessary that

$$3s^2[2j_0^{**} + (C-1)s] \equiv 0 \pmod{2n}. \qquad \text{(IV. 6)}$$

Again we require --in order to guarantee that the colored pattern resulting from the choice $j_0 = j_0^{**}$ be the same (aside from color) as the monochromatic pattern generated by choosing $j_0 = 1$ -- that

$$j_0^{**} = 1 + Zs \quad (Z = 0, 1, 2, \ldots). \qquad \text{(IV. 7)}$$

Substituting Eq. IV. 7 in Eq. IV. 6, we obtain

$$6s^3\ Z \equiv -\ 3s^2[(C-1)s+2] \pmod{2n}. \qquad \text{(IV. 8)}$$

---

*The Theory of Numbers, Emil Grosswald, Macmillan Co. (1966), pp. 46-47

For most parameter sets, obtaining values for j_0* and j_0** from the search algorithm based on Eqs. IV. 2, 4, 6, 8, in the compiled BASIC program for cubic K-patterns I have written for the IBM PC, requires at most 2-3 seconds of computation. Occasionally when n⋙1, the calculations take somewhat longer.

As an illustration of the use of Eq. IV. 8, consider the lattice pattern for ⟨n, s⟩ = ⟨13, 4⟩ (parity class f). Complexity for this case = 13. From Eq. IV. 8, we find that Z is a solution of the linear congruence

$$20Z \equiv -8 \pmod{26}. \qquad \text{(IV. 9)}$$

Hence Z = 10, 23, 36, 49 ... From Eq. IV. 7, we obtain the result j_0** = 15. The cubic residues for j = j_0**, j_0** + s, j_0** + 2s, ..., j_0** + (complexity - 1)s are therefore

[21 21 25 1 21 1 13 25 5 25 1 5 5]

(0) (4) (2) (20) (6) (12) (12) (6) (20) (2) (4) (0)

The residue differences are given in the lower row. It is easily verified that these difference terms are symmetrically distributed about the center (13) of the residue sequence. This is equivalent to the statement that changes in direction between consecutive unit vectors are symmetrically distributed, i.e., that locally perfect color reflection symmetry will result if a particular color is assigned to the unit vectors specified by this set of residues.

Now consider a second example: ⟨n, s⟩ = ⟨64, 10⟩ (parity class a). In this case, the symmetry is d_8, and complexity = 8. Here, Z is a solution of the linear congruence

$$112Z \equiv -96 \pmod{128}. \qquad \text{(IV. 10)}$$

Hence Z = 22, 54, 86 ... From Eq. IV. 7, j_0** = 93 or 29. The cubic residues for j = j_0**, j_0** + s, j_0** + 2 s, ..., j_0** + (complexity - 1)s (for either value of j_0**) are:

[5 119 81 3 125 47 9 123]

(114) (90) (50) (122) (50) (90) (114).

It is easily verified in this case also that the difference terms are symmetrically distributed about the center (3 - 125) of the residue sequence. Again this is equivalent to the statement that the direction changes between consecutive pattern vectors are symmetrical with respect to the color distribution.

To illustrate the use of Eqs. IV. 2 and IV. 4 to obtain a j_0* value for a K-pattern which belongs to a parity class lacking the Q property, let us first choose the example $\langle n, s\rangle = \langle 6, 1\rangle$, for which complexity = 4 and symmetry = d_3 (parity class c). From Eq. IV. 4, Z is a solution of

$$6Z \equiv -6 \pmod{12}. \qquad \text{(IV. 11)}$$

Hence Z = 1, 3, 5,..., and j_0* (from Eq. IV. 3) = 2, 4, 6, ... If we choose j_0* = 4, the cubic residues for j_0* - 3s, j_0* - 2s, j_0* - s, j_0*, j_0* + s, j_0* + 2s, and j_0* + 3s are:

[1 8 3 4 5 0 7]

(7) (7) (1) (1) (7) (7) ;

the difference terms appear below.

Finally, consider $\langle n, s\rangle = \langle 16, 5\rangle$; complexity = 32 and symmetry = lattice (L) (parity class d). From Eq. IV. 4, Z is a solution of

$$14Z \equiv -22 \pmod{32}.$$

Hence Z = 3, 19, 35,..., and j_0* (from Eq. IV. 3) = 16, 96, 176,...

If we choose j_0* = 96, the cubic residues for ... j_0* - 3s, j_0* - 2s, j_0* - s, j_0*, j_0* + s, j_0* + 2s, j_0* + 3s, ... are:

[... 17 24 3 0 29 8 15 ...]

... (7) (11)(29)(29) (11)(7)...;

again, the difference terms appear below.

K-patterns for both this example and the previous one are included among the illustrations of the sym-complexity parity classes on pp. 16a, b, and c.

Because it allows the calculation of j_0 from a single formula, it appears that Theorem 6 (p. 5) should be useful in the expression of color reflection symmetry, for any K-pattern which satisfies either conditions (a) or (b) of the theorem. Unfortunately, its usefulness is presently limited by the absence of complete information concerning the conditions under which a K-pattern based on a j_0 value calculated from the theorem is the same (aside from color) as the pattern obtained with $j_0 = 1$. Although I have not yet settled this question, it should not be an excessively time-consuming task to obtain a reliable solution by resorting to the same empirical methods I used earlier to determine the p-s rules for K-patterns with $2 \leq \alpha \leq 7$ (cf. § III A, p. 8).

When time permits, I will obtain such a solution. In the meantime, I have discovered, in the course of testing many examples with the aforementioned computer program, only two exceptions (described immediately below) to the "rule" which states that a value for j_0 obtained from Theorem 6, whenever the conditions of the theorem are satisfied, gives the same pattern (aside from color) as does $j_0 = 1$, in spite of the fact that the value of j_0 obtained from Theorem 6 only rarely belongs to the set of j-values $\{1 + Zs\}$.

These two exceptional cases are as follows:

Let p_1 and p_2 = odd primes $\neq 3$; $(p_1, p_2) = 1$.

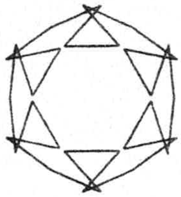
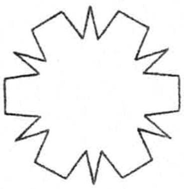
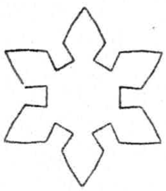
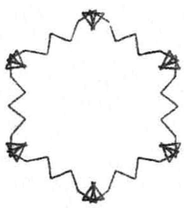
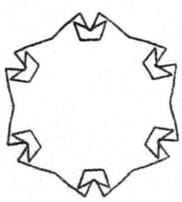
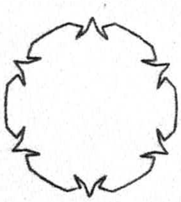
d_6 patterns: $\langle n, s \rangle = \langle 3^3 p_1, 3p_2 \rangle$; complexity = p_1.

d_3 patterns: $\langle n, s \rangle = \langle 3^3 p_1, 6p_2 \rangle$; complexity = p_1.

It is easy to verify (by substitution for n, s, and complexity in Eqs. II. 13 - 14) that Theorem 6a applies to the d_6 patterns, and Theorem 6b to the d_3 patterns. The patterns which result in each case from the use of a j_0 value obtained from Theorem 6 also have complexity = p_1, but the order of the symmetry group in each case is reduced by a factor of 3 as compared with the pattern obtained with $j_0 = 1$.

On the following page are shown six simple examples of d_6 patterns which belong to the $\langle 3^3p_1, 3p_2 \rangle$ family described above. A search algorithm which implements Eqs. IV. 6 and IV. 7 was used in the BASIC computer program to obtain the Z (and therefrom j_0**) values shown in the last column (although of course these values can be computed very quickly by hand).

SIX EXAMPLES OF d_6 PATTERNS OF THE $\langle 3^3 p_1, 3p_2 \rangle$ FAMILY

	n	s	$Z = (j_0^{**} - 1)/s$
	$3^3 \cdot 5 = 135$	$3 \cdot 7 = 21$	2
	$3^3 \cdot 5 = 135$	$3 \cdot 11 = 33$	1
	$3^3 \cdot 7 = 189$	$3 \cdot 5 = 15$	3
	$3^3 \cdot 11 = 297$	$3 \cdot 5 = 15$	3
	$3^3 \cdot 11 = 297$	$3 \cdot 13 = 39$	4
	$3^3 \cdot 11 = 297$	$3 \cdot 17 = 51$	9

V. *Lattice repeat distance* for K-patterns with lattice symmetry

Knowing nothing about the published literature on K-patterns, I have no idea to what extent any of the findings summarized in this note are already known*. Recalling several comments made to me two years ago by Andrew Odlyzko, however, I suppose that much is already known (and proved) concerning both cubic and quintic K-patterns, at least for the case $j_o = 0$. I'm certain Andrew mentioned that in the case of K-patterns with lattice symmetry, both for $\alpha = 3$ and $\alpha = 5$, expressions have been found (and also proved?) for the ***lattice repeat distance***, i.e., the distance between corresponding points of adjacent unit cells. I have not proved any results in this area, but I offer the following (presumably well-known) conjectures based on observation.

A. For K-patterns with lattice symmetry of the type described in §B. 2b (p. 9) (and I imagine other types as well), the lattice repeat distance is independent of s.

B. For cubic K-patterns with lattice symmetry, for which $n = 2^{\nu}$ ($\nu \not\equiv 0 \pmod 3$; $s = 1$; $j_o = 1$), the lattice repeat distance $= 2^{\delta}$, where $\delta = \nu - \text{int}(\nu/3)$.

C. For cubic K-patterns with lattice symmetry, for which $n = p^{\nu}$ (p = odd prime > 3; $\nu \not\equiv 1 \pmod 3$; s = even integer < period $\ni$ $(s, p) = 1$; $j_o = 1$), the lattice repeat distance $= p^{\nu - 1}$.

D. For cubic K-patterns with lattice symmetry, for which $n = 3^{\nu}$ ($\nu \not\equiv 1 \pmod 3$; s = even integer < period $\ni$ $(s, 3) = 1$; $j_o = 1$), the lattice repeat distance $= \beta \cdot 3^{\delta}$, where $\delta = \nu - 2 - \text{int}((\nu - 1)/3)$, and

$$\beta = 1 \text{ if } \nu \equiv 0 \pmod 3;$$

$$\beta = 2\cos(\pi/9) - 1 \text{ if } \nu \equiv 2 \pmod 3.$$

For $\alpha = 3$ and s = even integer, the first 13 primes p of the form $p \equiv 1 \pmod 3$ (cf. §III. B.2 on p. 9) generate lattice patterns with repeat distances which exhibit the following apparently irregular behavior:

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*I originally discovered K-patterns as the aftermath (sic) of a typo made in the course of keying in a computer program in a quite unrelated problem area. I like to think of this experience as an example of what the late Max Delbruck used to call ***controlled sloppiness.***

prime ≡ 1 (mod 3)	lattice repeat distance
7	4.406
13	6.953
19	5.855
31	10.252
37	7.561
43	9.665
61	13.358
67	14.947
73	13.454
79	7.325
97	19.620
103	4.661
109	0.668

For a prime p for which the lattice repeat distance << p, the steady state form of the pattern is not reached until several (overlapping) replicas of the unit cell have been drawn. Quite attractive patterns sometimes result in these cases. (No examples illustrating this phenomenon are included among the K-patterns illustrated in this note.)

Acknowledgements

I wish to thank Andrew Odlyzko for the interest and encouragement he expressed, soon after I began the investigation of K-patterns. I also thank Patrick Gallagher for his helpful comments and suggestions.

I gratefully acknowledge help received from my mathematician colleagues: Andy Earnest, Don Redmond, Robert Robinson, Phil Feinsilver, and Don Tveter. Andy steered me toward the valuable paper by A. E. Livingston and M. L. Livingston cited in §III. C. He and Don Redmond also (independently of one another) extended the analysis of pattern redundancy mentioned in §III. B.2 to the case of composite n. Robert first clarified for me the problem of pattern redundancy for prime n (cf. §III. B.2), pointing out the role of the cyclotomic equation. Phil provided encouragement, and he also helped me read the paper by Livingston and Livingston cited above. Don Tveter made a valuable suggestion about accurate calculation of power residues. Carl Pomerance wrote a timely article on "The Search for Prime Numbers" in Scientific American (December, 1982), which enabled me to refine and accelerate the calculation of power residues still further.

I wish especially to thank Edward Zhu for his help in writing improved FORTRAN programs for generating K-patterns as CalComp plots and as Ramtek displays, to supplement the Tektronix BASIC versions with which I began. I also thank him for his skillful instruction in the use of the IBM PC.

Arch Harris generously provided me with access to the SIU Computer Science Department's PDP/11 - supported Ramtek graphics console. Use of this fast interactive system was critically important in enabling me to get a glimpse of some of the systematic structure underlying the distribution of K-patterns.

Bibliography

Introduction to Geometry, H. S. M. Coxeter, John Wiley & Sons (1963), pp. 39-49.

Computer Recreations, Brian Hayes, Scientific American, Feb., 1984.

Color and Symmetry, Arthur L. Loeb, Wiley-Interscience (1971).

APPENDIX A

Proof that cubic K-patterns for which $\langle n, s\rangle = \langle 2^{\nu}, \text{odd integer}\rangle$ lack property Q

As stated in § IV, I have not proved that K-patterns which belong to the sym-complexity parity classes c and d (cf. table on p. 15) lack property Q (cf. p. 12). It is not difficult, however, to prove that for $\alpha = 3$, K-patterns corresponding to certain *sub-classes* of the parity classes c (odd-even) and d (1-even) lack property Q. These two sub-classes are defined by the parameter set $\langle n, s\rangle = \langle 2^{\nu}, \text{odd integer}\rangle$. I have found (but not proved) that for patterns with this parameter set,

(i) if $\nu \equiv 0 \pmod 3$, symmetry is d_1, and complexity is even.

(ii) if $\nu \not\equiv 0 \pmod 3$, symmetry is L, and complexity is even.

<u>Theorem</u>: If $n = 2^{\nu}$ $(\nu = 1, 2, 3, \ldots)$,

$s = 1, 3, 5, \ldots 2n - 1,$

$j_0 = 1,$

and

$\alpha = 3,$

then

$\exists$ integer W $(0<W<2n)$ $\ni$

(i) $(1 + Ws)^3 \equiv 0 \pmod{2n}$, (A.1)

and

(ii) $[1+(W+k)s]^3 - [1+Ws]^3 \equiv [1+Ws]^3 - [1+(W-k)s]^3 \pmod{2n}$ for all integer k. (A.2)

Eq. A.1 implies that the pattern vector $\vec{v}(W) = \exp[i(1+sW)^3\pi/n]$ is *horizontal*.

Eq. A.2 is the necessary and sufficient condition for the K-pattern for $\langle n,s\rangle$

to be symmetrical by reflection in a line which is the perpendicular bisector of the pattern vector $\vec{v}(W)$.

Proof:

It is trivial to prove that if

$$1 + Ws \equiv 0 \pmod{2n}, \qquad \text{(A.3)}$$

then both Eqs. A.1 and A.2 are satisfied.

To prove that $\exists$ a positive integer $W < 2n \ni$ Eq. A.3 is satisfied, it is only necessary to invoke the elementary theorem on linear congruences cited on p. 18. According to this theorem, since $(s, 2n) = 1$, Eq. A.3 has a unique solution (mod 2n). Since s is odd, and $2n = 2^{\nu+1}$ is even, W is of the form

$$W = -s^{n-1} + i \cdot 2n \qquad (i = 1,2,3,\ldots). \qquad \text{(A.4)}$$

That $W < 2n$ follows from the fact that W is odd and is a solution (*mod 2n*) of the congruence A.3.

Remark:

Note that for $s' = s + 2n$,

$$(1 + Ws')^3 \not\equiv (1 + Ws)^3 \pmod{2n}. \qquad \text{(A.5)}$$

Hence it is not necessary to consider values of $s > 2n - 1$.

APPENDIX B

COLOR REFLECTION SYMMETRY FOR ALL ODD $\alpha \geq 3$

Let us now examine the problem of color reflection symmetry for all odd $\alpha \geq 3$, both for patterns which have property Q and for those which do not (cf. pp. 12-17).

1. K-patterns which do not have property Q

Let us simplify the notation slightly by writing j_0 instead of j_0^*. Then the analog of Eq. IV.1 for any odd α is

$$(j_0+ks)^3-j_0^3+(j_0-ks)^3-j_0^3 \equiv 0 \pmod{2n}; \tag{B.1}$$

$$k = 0,1,2,\ldots,C/2. \tag{B.2}$$

Eqs. B.1-2 state that if the pattern vector for j_0 is perpendicularly bisected by a line of reflection R, then the angle distribution for all vectors in the sector is symmetrical by reflection in R. After cancelling terms in Eq. B.1, we obtain

$$2\sum_{i=1}^{(\alpha-1)/2} \binom{\alpha}{2i} j_0^{\alpha-2i}(ks)^{2i} \equiv 0 \pmod{2n}. \tag{B.3}$$

Now if we let

$$\gamma = \text{g.c.d.}\left(\binom{\alpha}{2},\binom{\alpha}{4},\ldots,\binom{\alpha}{\alpha-1}\right), \tag{B.4}$$

then

$$\boxed{2\gamma s^2 j_0 \equiv 0 \pmod{2n}, \text{ for all odd } \alpha \geq 3} \tag{B.5}$$

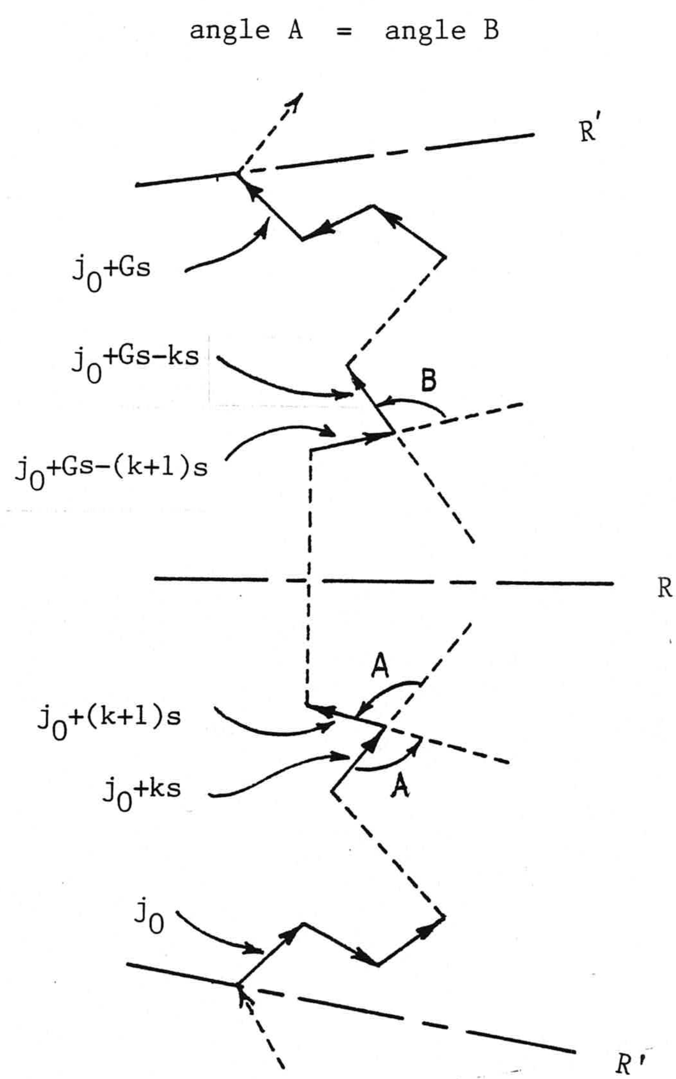
is a sufficient condition for the congruence B.1 to be satisfied. For $\alpha = 3$, Eq. B.5 yields Eq. IV.2, since for α = prime,

$$\alpha \mid \binom{\alpha}{k} \tag{B.6}$$

for $1 \leq k \leq \alpha-1$ (cf. p.32). It is almost certainly true that the congruence B.5 is also the necessary condition for Eq. B.1, but to prove that it is would require invoking theorems on solutions of polynomial congruences with which I have a barely nodding acquaintance.

2. K-patterns which have property Q

Now consider the more complicated problem of deriving the conditions for color reflection symmetry for K-patterns for $\alpha \geq 3$ which have the property Q (cf. pp. 12 ff.). In the sketch at the right, a pair of consecutive pattern vectors and their images under reflection (in R) are shown in one sector of a centrosymmetric K-pattern with dihedral symmetry and property Q. The value of the pattern index j is indicated for representative vectors. The direction of a vector with index j is $j^{\alpha}\pi/n$ (mod 2n).



The polynomial congruence which expresses the necessary condition for color reflection symmetry is equivalent to the statement that angle A = angle B. (Eq. IV.5 is the special case of this congruence for $\alpha = 3$.) Now let G = C-1. Then the equality of the angles A and B is expressed by setting

$$Z_{\alpha}(j_0,G,s,k) \equiv 0 \pmod{2n}, \tag{B.7}$$

where

$$Z_{\alpha} = [j_0+(k+1)s]^{\alpha}-[j_0+ks]^{\alpha}+[j_0+Gs-(k+1)s]^{\alpha}-[j_0+Gs-ks]^{\alpha} ; \tag{B.8}$$

$k = 0,1,2,\ldots,k_{max}$;

$k_{max} = (C-3)/2$ if C is odd;

$= (C-4)/2$ if C is even.

After expanding Eq. B.8 and then eliminating cancelled terms, we obtain

$$Z_\alpha = \binom{\alpha}{1} s\ [(j_0+ks)^{\alpha-1} - \{(j_0+Gs)-ks\}^{\alpha-1}]$$

$$+\binom{\alpha}{2} s^2[(j_0+ks)^{\alpha-2} + \{(j_0+Gs)-ks\}^{\alpha-2}]$$

$$+\binom{\alpha}{3} s^3[(j_0+ks)^{\alpha-3} - \{(j_0+Gs)-ks\}^{\alpha-3}]$$

$$+\binom{\alpha}{4} s^4[(j_0+ks)^{\alpha-4} + \{(j_0+Gs)-ks\}^{\alpha-4}]$$

$$+ \dots$$

$$+\binom{\alpha}{\alpha-2} s^{\alpha-2}[(j_0+ks)^2 - \{(j_0+Gs)-ks\}^2]$$

$$+\binom{\alpha}{\alpha-1} s^{\alpha-1}[(j_0+ks) + \{(j_0+Gs)-ks\}]. \tag{B.9}$$

If α is a prime, then each binomial coefficient $\binom{\alpha}{k}$ is divisible by α, for $1 \leq k \leq \alpha-1$, according to an elementary theorem*. Hence in this case, $\alpha | Z_\alpha$, and we may conveniently write

$$Z_\alpha = \alpha Z'_\alpha$$
$$\equiv 0 \pmod{2n}. \tag{B.10}$$

For any odd α, not necessarily prime, let

$$\beta = \text{g.c.d.}\left(\binom{\alpha}{1}, \binom{\alpha}{2}, \dots, \binom{\alpha}{\alpha-1}\right). \tag{B.11}$$

Then for every odd α, $\beta | Z_\alpha$; we write

$$Z_\alpha = \beta Z'$$
$$\equiv 0 \pmod{2n}. \tag{B.12}$$

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*Elements of Number Theory, K. Ireland and M. I. Rosen, Bogden and Quigley, Inc. (1972), p. 43

For the first three non-prime odd α -- 9, 15, and 21 -- β = 3, 1, and 1, respectively. When α is prime, $\beta = \alpha$.

Now let us express Z_α and Z'_α as follows:

$$Z_\alpha = Z_{\alpha,0} + Z_{\alpha,k} \tag{B.13}$$

and

$$Z'_\alpha = Z'_{\alpha,0} + Z'_{\alpha,k} \,. \tag{B.14}$$

$Z_{\alpha,0}$ and $Z'_{\alpha,0}$ are independent of k, while $Z_{\alpha,k}$ and $Z'_{\alpha,k}$ are polynomials in k. It can easily be proved from Eq. B.9 that $Z_{\alpha,k}$ (and therefore also $Z'_{\alpha,k}$) is of degree $\alpha-2$.

I have derived expressions for $Z_{\alpha,0}$ (see below) for α = 3, 5, 7, and 9. I have so far obtained expressions for $Z_{\alpha,k}$ only for $\alpha = 3$ and $\alpha = 5$. (The calculations for $Z_{\alpha,k}$, when done by hand, become quite lengthy, beginning with $\alpha = 7$.)

For both $\alpha = 3$ and $\alpha = 5$, it is found that

$$\beta s^2(2j_0+Gs) \mid Z_\alpha \,. \tag{B.15}$$

For all four of these cases -- α = 3, 5, 7, and 9 --

$$\beta s^2(2j_0+Gs) \mid Z'_{\alpha,0} \,. \tag{B.16}$$

A plausible conjecture seems to be that

$$\beta s^2(2j_0+Gs) \mid Z_\alpha \text{ for all odd } \alpha \geq 3. \tag{B.17}$$

If Eq. B.17 is correct, then a sufficient condition for Eq. B.7 is

$$\boxed{\beta s^2(2j_0+Gs) \equiv 0 \pmod{2n}, \text{ for all odd } \alpha \geq 3.} \tag{B.18}$$

The expressions obtained thus far for $Z_{\alpha,0}$ and $Z_{\alpha,k}$ are as follows:

$\underline{\alpha = 3}$:

$$Z_{3,0} = 3s^2(2j_0+Gs)\{1-G\} ; \tag{B.19}$$

$$Z_{3,k} = 3s^2(2j_0+Gs)\{2k\} . \tag{B.20}$$

$\underline{\alpha = 5}$:

$$Z_{5,0} = 5s^2(2j_0+Gs)\{(1-G)[2j_0(j_0+Gs)+(1-G+G^2)s^2]\}; \tag{B.21}$$

$$\begin{aligned} Z_{5,k} = 5s^2(2j_0+Gs)\{&k[2(2j_0(j_0+Gs)+(2-3G+2G^2)s^2)] \\ &+k^2[6(1-G)s^4] \\ &+k^3[4s^2]\} . \end{aligned} \tag{B.22}$$

$\underline{\alpha = 7}$:

$$\begin{aligned} Z_{7,0} = 7s^2(2j_0+Gs)\{&3(1+G)j_0^4 \\ &+(1-G)[6Gsj_0^3 \\ &\qquad +(7G^2-5G+5)s^2j_0^2 \\ &\qquad +G(4G^2-5G+5)s^3j_0 \\ &\qquad +(G^4-2G^3+3G^2-2G+1)s^4]\} . \end{aligned} \tag{B.23}$$

$Z_{7,k}$ = polynomial in k of degree 5, which includes terms in

k, k^2, k^3, k^4, and k^5. (B.24)

$\alpha = 9$:

$$Z_{9,0} = 3s^2(2j_0+Gs)\{(1-G)[12j_0{}^6 +36Gsj_0{}^5 +6(11G^2-7G+7)s^2j_0{}^4 +12G(6G^2-7G+7)s^3j_0{}^3 +4(12G^4-21G^3-28G^2-14G+7)s^4j_0{}^2 +2G(9G^4-21G^3+35G^2-28G+14)s^5j_0 +(9G^6-9G^5+19G^4-23G^3-19G^2-19G+3)s^6]\}. \quad (B.25)$$

$Z_{9,k}$ = polynomial in k of degree 7, which includes terms in k, k^2, k^3, k^4, k^5, k^6, and k^7. (B.26)

It is obviously desirable that the algorithm used to express color reflection symmetry be as simple as possible, in order to maximize program execution speed. If it can be proved that the linear congruence of Eq. B.18 provides valid solutions for j_0 for all odd α, then the time required to search for a proper Z value for any K-pattern with a reasonable value for complexity ($C \lessapprox 1000$) will never be excessively long. Using a compiled version of the IBM PC program for cubic K-patterns, I have found that with a search algorithm which tests consecutive Z values 0,1,2,... until one is found which satisfies Eq. IV.8, the search time per Z value is less than 80 msec. Consequently, even if more than 100 candidate values are tested before a solution is found, the search time never exceeds 8 seconds.

Finally, consider the extension of Theorems 6a and 6b to all $\alpha \geq 3$. Let us assume that Eq. B.18 is the correct statement of the necessary condition for color reflection symmetry. Then we conclude that color reflection symmetry results if either

a) n and s are both even or both odd,

$\beta s^2(n+Cs) \equiv 0 \pmod{2n}$,

and $j_0 = (n+s)/2$; (B.27)

or b) s is even,

$\beta s^3 C \equiv 0 \pmod{2n}$,

and $j_0 = s/2$. (B.28)

Still unresolved, however, is the problem of specifying which K-patterns, for any given value of α, are geometrically the same -- aside from color -- whether they are computed with $j_0 = 1$ or with j_0 = a value obtained from Eqs. B.27-28.